The ideal-valued index for a dihedral group action, and mass partition by two hyperplanes

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July 21, 2008

Abstract

We compute the complete Fadell–Husseini index of the dihedral group $D_8 = (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2$ acting on $S^d \times S^d$ for $\mathbb{F}_2$ and for $\mathbb{Z}$ coefficients, that is, the kernels of the maps in equivariant cohomology

$$H^*_{D_8}(pt, \mathbb{F}_2) \quad\longrightarrow\quad H^*_{D_8}(S^d \times S^d, \mathbb{F}_2)$$

and

$$H^*_{D_8}(pt, \mathbb{Z}) \quad\longrightarrow\quad H^*_{D_8}(S^d \times S^d, \mathbb{Z}).$$

This establishes the complete cohomological lower bounds, with $\mathbb{F}_2$ and with $\mathbb{Z}$ coefficients, for the two hyperplane case of Grünbaum's 1960 mass partition problem: For which d and j can any j arbitrary measures be cut into four equal parts each by two suitably-chosen hyperplanes in $\mathbb{R}^d$? In both cases, we verify that the ideal bounds are not stronger than previously established bounds based on one of the maximal abelian subgroups of D_8.

Contents

*Supported by the grant 144018 of the Serbian Ministry of Science and Technological development
**Partially supported by DFG

1 Introduction

1.1 The hyperplane mass partition problem

A *mass distribution* on $\mathbb{R}^d$ is a finite Borel measure $\mu(X) = \int_X f d\mu$ determined by an integrable density function $f : \mathbb{R}^d \to \mathbb{R}$.

Every affine hyperplane $H = \{x \in \mathbb{R}^d \mid \langle x, v \rangle = \alpha\}$ in $\mathbb{R}^d$ determines two open halfspaces

$$H^- = \{x \in \mathbb{R}^d \mid \langle x, v \rangle < \alpha\} \text{ and } H^+ = \{x \in \mathbb{R}^d \mid \langle x, v \rangle > \alpha\}.$$

An *orthant* of an arrangement of k hyperplanes $\mathcal{H} = \{H_1, H_2, ..., H_k\}$ in $\mathbb{R}^d$ is an intersection of halfspaces $\mathcal{O} = H_1^{\alpha_1} \cap ... \cap H_k^{\alpha_k}$, for some $\alpha_j \in \mathbb{Z}_2$. Thus there are 2^k orthants determined by $\mathcal{H}$ and they are naturally indexed by elements of the group $(\mathbb{Z}_2)^k$.

An arrangement of hyperplanes $\mathcal{H}$ *equiparts* a collection of mass distributions $\mathcal{M}$ in $\mathbb{R}^d$ if for each orthant $\mathcal{O}$ and each measure $\mu \in \mathcal{M}$ we get

$$\mu(\mathcal{O}) = \tfrac{1}{2^k}\mu(\mathbb{R}^d).$$

A triple of integers (d, j, k) is *admissible* if for every collection $\mathcal{M}$ of j mass distributions in $\mathbb{R}^d$ there exists an arrangement of k hyperplanes $\mathcal{H}$ equiparting them.

The general problem formulated by Grünbaum [13] in 1960 can be stated as follows.

Problem 1.1. Determine the function $\Delta : \mathbb{N}^2 \to \mathbb{N}$ given by

$$\Delta(j,k) = \min\{d \mid (d,j,k) \text{ is an admissible triple}\}.$$

The case of one hyperplane, $\Delta(j,1) = j$, is the famous ham sandwich theorem, which is equivalent to the Borsuk–Ulam theorem. The equality $\Delta(2,2) = 3$, and consequently $\Delta(1,3) = 3$, was proven by Hadwiger [14]. Ramos [24] gave a general lower bound for the function Δ,

$$\Delta(j,k) \geq \tfrac{2^k-1}{k} j. \tag{1}$$

Recently, Mani, Vrećica and Živaljević [21] applied Fadell–Husseini index theory for an elementary abelian subgroup of $H_1 \subseteq D_8$ to get a new upper bound for Δ,

$$\Delta(2^q + r, k) \leq 2^{k+q-1} + r. \tag{2}$$

In the case of $j = 2^{l+1} - 1$ measures and $k = 2$ hyperplanes these bounds yield the equality

$$\Delta(j,2) = \lceil \tfrac{3}{2} j \rceil.$$

1.2 Statement of the main result ($k = 2$, indices)

This paper addresses Problem 1.1 for $k = 2$ using two different but related Configuration Space/Test Map schemes (Section 2, Proposition 2.2).

- The **product scheme** is the classical one, already considered in [28] and [21]. The problem is translated to the problem of the existence of a W_k-equivariant map,

  $$Y_{d,k} := \left(S^d\right)^k \longrightarrow S\left(\left(R_{2^k}\right)^j\right),$$

  where $W_k = (\mathbb{Z}_2)^k \rtimes S_k$ is the Weyl group.

- The **join scheme** is a new one. It connects the problem with classical Borsuk–Ulam properties in the spirit of Marzantowicz [22]. Is there a W_k-equivariant map

  $$X_{d,k} := \left(S^d\right)^{*k} \longrightarrow S\left(U_k \times \left(R_{2^k}\right)^j\right)?$$

  The W_k-representations R_{2^k} and U_k are introduced in Section 2.2.

Obstruction theory methods cannot be applied to either scheme directly for $k > 1$, since the W_k-actions on the respective configuration spaces $\left(S^d\right)^k$ and $\left(S^d\right)^{*k}$ are *not free* (compare [21, Section 2.3.3], assumptions on the manifold M^n). Therefore we analyze the associated equivariant question for $k = 2$ via the Fadell–Husseini ideal index theory method. We show that the join scheme considered from the Fadell–Husseini point of view, with either $\mathbb{F}_2$ or $\mathbb{Z}$ coefficients, does not lead to any improvement compared to the known bounds (Remarks 5.3 and 6.2). In the case of the product scheme we give the new and best possible ideal bounds by proving the following theorem.

Theorem 1.2. *Let π_d, $d \geq 0$, be polynomials in $\mathbb{F}_2[y,w]$ given by*

$$\pi_d(y,w) = \sum_i \binom{d-1-i}{i}_{\bmod 2} w^i y^{d-2i}$$

and Π_d, $d \geq 0$, be polynomials in $\mathbb{Z}[\mathcal{Y},\mathcal{M},\mathcal{W}]/\langle 2\mathcal{Y}, 2\mathcal{M}, 4\mathcal{W}, \mathcal{M}^2 - \mathcal{W}\mathcal{Y}\rangle$ given by

$$\Pi_d(\mathcal{Y},\mathcal{W}) = \sum_i \binom{d-1-i}{i}_{\bmod 2} \mathcal{W}^i \mathcal{Y}^{d-2i}.$$

(A) *$\mathbb{F}_2$-bound: The triple $(d,j,2) \in \mathbb{N}^3$ is admissible if*

$$y^j w^j \notin \langle \pi_{d+1}, \pi_{d+2} \rangle \subseteq \mathbb{F}_2[y,w].$$

(B) $\mathbb{Z}$*-bound: The triple* $(d,j,2)\in\mathbb{N}^3$ *is admissible if*

$$\left\langle \begin{matrix} (j-1)_{\mathrm{mod}\,2}\;\mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}}, & \\ j_{\mathrm{mod}\,2}\;\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M}, & j_{\mathrm{mod}\,2}\;\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}} \end{matrix} \right\rangle \subseteq \left\langle \begin{matrix} (d-1)_{\mathrm{mod}\,2}\;\Pi_{\frac{d+2}{2}}, & (d-1)_{\mathrm{mod}\,2}\;\Pi_{\frac{d+4}{2}}, \\ (d-1)_{\mathrm{mod}\,2}\;\mathcal{M}\Pi_{\frac{d}{2}}, & \\ d_{\mathrm{mod}\,2}\;\Pi_{\frac{d+1}{2}}, & d_{\mathrm{mod}\,2}\;\Pi_{\frac{d+3}{2}} \end{matrix} \right\rangle$$

in the ring $\mathbb{Z}[\mathcal{Y},\mathcal{M},\mathcal{W}]/\langle 2\mathcal{Y},2\mathcal{M},4\mathcal{W},\mathcal{M}^2-\mathcal{W}\mathcal{Y}\rangle$.

Remark 1.3. Let $\hat{\Pi}_d$, $d\geq 0$, be the sequence of polynomials in $\mathbb{Z}[Y,W]$ defined by $\hat{\Pi}_0=0$, $\hat{\Pi}_1=Y$ and $\hat{\Pi}_{d+1}=Y\hat{\Pi}_d+W\hat{\Pi}_{d-1}$ for $d\geq 2$. Then the sequences of polynomials Π_d and π_d are reductions of the polynomials $\hat{\Pi}_d$. The polynomials $\hat{\Pi}_d$ can be also described by the generating function [25] (formal power series)

$$\sum_{d\geq 0}\hat{\Pi}_d=\frac{Y}{1-Y-W}$$

where $\hat{\Pi}_d$ is homogeneous of degree $2d$ if we set $\deg(Y)=2$ and $\deg(W)=4$.

Theorem 1.2 is a consequence of a topological result, the complete and explicit computation of the relevant Fadell–Husseini indices of the D_8-space $S^d\times S^d$ and the D_8-sphere $S(R_4^{\oplus j})$.

Theorem 1.4.

(A) $\mathrm{Index}^{3j}_{D_8,\mathbb{F}_2}S(R_4^{\oplus j})=\mathrm{Index}_{D_8,\mathbb{F}_2}S(R_4^{\oplus j})=\langle y^jw^j\rangle$

(B) $\mathrm{Index}^{d+2}_{D_8,\mathbb{F}_2}(S^d\times S^d)=\langle\pi_{d+1},\pi_{d+2}\rangle$.

(C) $\mathrm{Index}^{3j+1}_{D_8,\mathbb{Z}}S(R_4^{\oplus j})=\begin{cases}\langle\mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}}\rangle, & \textit{for } j \textit{ even}\\ \langle\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M},\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}}\rangle, & \textit{for } j \textit{ odd}\end{cases}.$

(D) $\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}}S^d\times S^d=\begin{cases}\langle\Pi_{\frac{d+2}{2}},\Pi_{\frac{d+4}{2}},\mathcal{M}\Pi_{\frac{d}{2}}\rangle, & \textit{for } d \textit{ even}\\ \langle\Pi_{\frac{d+1}{2}},\Pi_{\frac{d+3}{2}}\rangle, & \textit{for } d \textit{ odd}\end{cases}.$

The sequence of Fadell–Husseini indexes will be introduced in Section 3. The actions of the dihedral group D_8 and the definition of the representation space $R_4^{\oplus j}$ are given in Section 2. Even though it does not seem to have any relevance to our study of Problem 1.1, the complete index $\mathrm{Index}_{D_8,\mathbb{F}_2}(S^d\times S^d)$ will also be computed in the case of $\mathbb{F}_2$ coefficients,

$$\mathrm{Index}_{D_8,\mathbb{F}_2}(S^d\times S^d)=\langle\pi_{d+1},\pi_{d+2},w^{d+1}\rangle. \tag{3}$$

1.3 Proof overview

The Problem 1.1 about mass partitions by hyperplanes can be connected with the problem of the existence of equivariant maps as discussed in Section 2, Proposition 2.2. The topological problems we face, about the existence of $W_k=(\mathbb{Z}_2)^k\rtimes S_k$-maps, for the product / join schemes,

$$\left(S^d\right)^k_{W_k}\longrightarrow S\left(R_{2^k}^{\oplus j}\right),\qquad \left(S^d\right)^{*k}\longrightarrow_{W_k}S\left(U_k\times R_{2^k}^{\oplus j}\right),$$

have to be treated with care because the actions of the Weyl groups W_k are not free. Note that there is no naive Borsuk–Ulam theorem for fixed point free actions. Indeed, in the case $k=2$ when $W_2=D_8$ there exists a W_2-equivariant map [4, Theorem 3.22, page 49]

$$S\left((V_{+-}\oplus V_{-+})^{10}\right)\longrightarrow_{W_2}S\left((U_2\oplus V_{--})^{8}\right)$$

where $\dim\left(V_{+-}\oplus V_{-+}\right)^{10}>\dim\left(U_2\oplus V_{--}\right)^8$. The $W_2=D_8$-representations $V_{+-}\oplus V_{-+}$, V_{--} and U_2 are introduced in Section 2.2.

In this paper we focus on the case of $k=2$ hyperplanes. Theorem 1.2 gives the best possible answer to the question about the existence of $W_2 = D_8$-maps

$$S^d \times S^d \longrightarrow S(R_4^{\oplus j})$$

from the point of view of Fadell–Husseini index theory (Section 3). We explicitly compute the relevant Fadell–Husseini indexes with $\mathbb{F}_2$ and $\mathbb{Z}$ coefficients (Thereom 1.4, Sections 5, 6, 7 and 8). Then Theorem 1.2 is a consequence of the basic index property, Proposition 3.2.

The index of the sphere $S(R_4^{\oplus j})$, with $\mathbb{F}_2$ coefficients, is computed in Section 5 by

- decomposition of the D_8-representation $R_4^{\oplus j}$ into a sum of irreducible ones, and
- computation of indexes of spheres of all irreducible D_8-representations.

The main technical tool is the restriction diagram derived in Section 4.3.2, which connects the indexes of the subgroups of D_8.

The index with $\mathbb{Z}$ coefficients is computed in Section 6 using

- (for j even) the results for $\mathbb{F}_2$ coefficients and comparison of Serre spectral sequences, and
- (for j odd) the Bockstein spectral sequence combined with known results for $\mathbb{F}_2$ coefficients and comparison of Serre spectral sequences.

The index of the product $S^d \times S^d$ is computed in Sections 7 and 8 by an explicit study of the Serre spectral sequence associated with the fibration

$$S^d \times S^d \to \mathrm{E}D_8 \times_{D_8} (S^d \times S^d) \to \mathrm{B}D_8.$$

The major difficulty comes from non-triviality of the local coefficients in the Serre spectral sequence. The computation of the spectral sequence with non-trivial local coefficients is done by an independent study of $H^*(D_8,\mathbb{F}_2)$-module and $H^*(D_8,\mathbb{Z})$-module structures of relevant rows in the Serre spectral sequence (Sections 7.1 and 8.1).

1.4 Evaluation of the index bounds

1.4.1 $\mathbb{F}_2$-evaluation

It was pointed out by Vrećica [26] that, with $\mathbb{F}_2$-coefficients, the D_8 index bound gives the same bounds as the $H_1 = (\mathbb{Z}_2)^2$ index bound. This observation follows from the implication

$$a^j b^j (a+b)^j \in \langle a^{d+1}, (a+b)^{d+1}\rangle \;\Rightarrow\; a^j b^j (a+b)^j \in \langle a^{d+1} + (a+b)^{d+1}, a^{d+2} + (a+b)^{d+2}\rangle.$$

By introducing a new variable $c := a+b$, it is enough to prove the implication

$$a^j c^j (a+c)^j \in \langle a^{d+1}, c^{d+1}\rangle \;\Rightarrow\; a^j c^j (a+c)^j \in \langle a^{d+1} + c^{d+1}, a^{d+2} + c^{d+2}\rangle. \tag{4}$$

Let us assume that $a^j c^j (a+c)^j \in \langle a^{d+1}, c^{d+1}\rangle$. The monomials in the expansion of $a^j c^j (a+c)^j$ always come in pairs

$$a^{d+k} c^{3j-d-k} + c^{d+k} a^{3j-d-k}.$$

This is also true when j is even since $\binom{j}{j/2} =_{\mathrm{mod}2} 0$ implies there are no middle term. The sequence of equations

$$\begin{array}{lcl}
a^{d+1}c^{3j-d-1} + c^{d+1}a^{3j-d-1} & = & (a^{d+1}+c^{d+1})(c^{3j-d-1}+a^{3j-d-1}) + a^{3j} + c^{3j} \\
a^{d+2}c^{3j-d-2} + c^{d+2}a^{3j-d-2} & = & (a^{d+1}+c^{d+1})(ac^{3j-d-2}+a^{3j-d-2}c) + a^{3j-1}c + ac^{3j-1} \\
\dots\dots & & \\
a^{3j} + c^{3j} & = & (a^{d+2}+c^{d+2})(a^{3j-d-2}+c^{3j-d-2}) + a^{d+2}c^{3j-d-2} + c^{d+2}a^{3j-d-2}
\end{array}$$

shows that all the binomials

$$a^{d+1}c^{3j-d-1}+c^{d+1}a^{3j-d-1}, \quad a^{d+2}c^{3j-d-2}+c^{d+2}a^{3j-d-2}, \quad \dots \quad , a^{3j}+c^{3j}$$

belong to the ideal $\langle a^{d+1}+c^{d+1}, a^{d+2}+c^{d+2}\rangle$ or none of them do.
Since for $3j-d-1$ even

$$\begin{array}{rcl} a^{d+1+\frac{3j-d-1}{2}}c^{\frac{3j-d-1}{2}}+c^{d+1+\frac{3j-d-1}{2}}a^{\frac{3j-d-1}{2}} & = & (a^{d+1}+c^{d+1})a^{\frac{3j-d-1}{2}}c^{\frac{3j-d-1}{2}} \\ & \in & \langle a^{d+1}+c^{d+1}, a^{d+2}+c^{d+2}\rangle \end{array},$$

and for $3j-d-1$ odd

$$\begin{array}{rcl} a^{d+2+\frac{3j-d-2}{2}}c^{\frac{3j-d-2}{2}}+c^{d+2+\frac{3j-d-2}{2}}a^{\frac{3j-d-2}{2}} & = & (a^{d+2}+c^{d+2})a^{\frac{3j-d-2}{2}}c^{\frac{3j-d-2}{2}} \\ & \in & \langle a^{d+1}+c^{d+1}, a^{d+2}+c^{d+2}\rangle \end{array},$$

the implication (4) is proved.

1.4.2 $\mathbb{Z}$-evaluation

More is true, even the complete D_8 index bound, now with $\mathbb{Z}$-coefficients, implies the same bounds for the $k=2$ hyperplanes mass partition problem.

Lemma 1.5. *Let* $a=\sum_{i=1}^{k}a_i 2^i$ *and* $b=\sum_{i=1}^{k}b_i 2^i$ *be the dyadic expansions. Then*

$$\binom{b}{a}_{\mathrm{mod}\,2}=\prod_{i=1}^{k}\binom{b_i}{a_i}_{\mathrm{mod}\,2}.$$

This classical fact [20] about binomial coefficients mod 2 yields the following property for the sequence of polynomials Π_d, $d\geq 0$.

Lemma 1.6. *Let* $q>0$ *and* i *be integers. Then*
(A) $\binom{2^q-1-i}{i}=\begin{cases}0, & i\neq 0\\ 1, & i=0\end{cases}$
(B) $\Pi_{2^q}=\mathcal{Y}^{2^q}$

Proof. The statement (B) is the direct consequence of the fact (A) and the definition of polynomials Π_d. For $i\notin\{1,\dots,2^{q-1}\}$ the statement (A) is true from boundary conditions on binomial coefficients. Let $i\in\{1,\dots,2^{q-1}\}$ and $i=\sum_{k\in I\subseteq\{0,\dots,q-1\}}2^k$. Then

$$2^q-1-i \quad = \quad 2^0+2^1+2^2+\dots+2^{q-1}-\sum_{k\in I\subseteq\{0,\dots,q-1\}}2^k \quad = \quad \sum_{k\in I^c\subseteq\{0,\dots,q-1\}}2^k$$

where I^c is the complementary index set in $\{0,\dots,q-1\}$. The statement (A) follows from Lemma 1.5 □

Let j be an integer such that $j=2^q+r$ where $0\leq r<2^q$ and $d=2^{q+1}+r-1$. Let us introduce the following ideals

$$A_j=\begin{cases}\langle\mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}}\rangle, & \text{for } j \text{ even}\\ \langle\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M},\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}}\rangle, & \text{for } j \text{ odd}\end{cases} \quad \text{and} \quad B_d=\begin{cases}\langle\Pi_{\frac{d+2}{2}},\Pi_{\frac{d+4}{2}},\mathcal{M}\Pi_{\frac{d}{2}}\rangle, & \text{for } d \text{ even}\\ \langle\Pi_{\frac{d+1}{2}},\Pi_{\frac{d+3}{2}}\rangle, & \text{for } d \text{ odd}\end{cases}$$

The fact that D_8 index bound with $\mathbb{Z}$-coefficients does not improve the mass partition bounds is the consequence of the following facts:

- $r=0 \;\Rightarrow\; A_j\subseteq B_d$,
- $(r\neq 2^q-1 \text{ and } A_j\subseteq B_d) \implies A_{j+1}\subseteq B_{d+1}$.

Lemma 1.7. $\langle \mathcal{Y}^{2^{q-1}}\mathcal{W}^{2^{q-1}}\rangle = A_{2^q} \subseteq B_{2^{q+1}-1} = \langle \Pi_{2^q}, \Pi_{2^q+1}\rangle$.

Proof. Since $\mathcal{Y}^{2^{q-1}} = \Pi_{2^{q-1}}$ by Lemma 1.6,

$$\mathcal{Y}^{2^{q-1}}\mathcal{W} \;=\; \Pi_{2^{q-1}}\mathcal{W} \;=\; \Pi_{2^{q-1}+2} + \mathcal{Y}\Pi_{2^{q-1}+1} \;\in\; \langle \Pi_{2^{q-1}+1}, \Pi_{2^{q-1}+2}\rangle.$$

By induction on the power i of $\mathcal{W}$ in $\mathcal{Y}^{2^{q-1}}\mathcal{W}^{2i}$,

$$\mathcal{Y}^{2^{q-1}}\mathcal{W}^{i} \in \langle \Pi_{2^{q-1}+i}, \Pi_{2^{q-1}+i+1}\rangle,$$

and consequently

$$\mathcal{Y}^{2^{q-1}}\mathcal{W}^{2^{q-1}} \in \langle \Pi_{2^q}, \Pi_{2^q+1}\rangle.$$

$\square$

Lemma 1.8. *If $r \neq 2^q - 1$ and $A_j \subseteq B_d$ then $A_{j+1} \subseteq B_{d+1}$.*

Proof. We distinguish two cases depending on the parity of j.

(A) Let j be even and $\mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}} \in \langle \Pi_{\frac{d+1}{2}}, \Pi_{\frac{d+3}{2}}\rangle$. There are polynomials α and β such that

$$\mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}} = \alpha\Pi_{\frac{d+1}{2}} + \beta\Pi_{\frac{d+3}{2}}.$$

Then

$$\begin{aligned}\mathcal{Y}^{\frac{(j+1)+1}{2}}\mathcal{W}^{\frac{(j+1)-1}{2}}\mathcal{M} \;&=\; \mathcal{Y}^{\frac{j+2}{2}}\mathcal{W}^{\frac{j}{2}}\mathcal{M} = \mathcal{Y}\mathcal{M}\left(\alpha\Pi_{\frac{d+1}{2}} + \beta\Pi_{\frac{d+3}{2}}\right)\\ &\in\; \langle \Pi_{\frac{(d+1)+2}{2}}, \mathcal{M}\Pi_{\frac{d+1}{2}}\rangle \subseteq \langle \Pi_{\frac{d+3}{2}}, \Pi_{\frac{d+5}{2}}, \mathcal{M}\Pi_{\frac{d+1}{2}}\rangle = B_{d+1},\end{aligned}$$

and

$$\begin{aligned}\mathcal{Y}^{\frac{(j+1)+1}{2}}\mathcal{W}^{\frac{(j+1)+1}{2}} \;&=\; \mathcal{Y}\mathcal{W}\left(\mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}}\right) = \mathcal{Y}\mathcal{W}\left(\alpha\Pi_{\frac{d+1}{2}} + \beta\Pi_{\frac{d+3}{2}}\right) = \alpha\mathcal{M}^2\Pi_{\frac{d+1}{2}} + \beta\mathcal{Y}\mathcal{W}\Pi_{\frac{d+3}{2}}\\ &\in\; \langle \mathcal{M}\Pi_{\frac{d+1}{2}}, \Pi_{\frac{d+3}{2}}\rangle \subseteq \langle \Pi_{\frac{d+3}{2}}, \Pi_{\frac{d+5}{2}}, \mathcal{M}\Pi_{\frac{d+1}{2}}\rangle = B_{d+1}.\end{aligned}$$

Thus $A_{j+1} \subseteq B_{d+1}$.

(B) Let j be odd and

$$\langle \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M}, \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}}\rangle = A_j \subseteq B_d = \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}, \mathcal{M}\Pi_{\frac{d}{2}}\rangle.$$

There are polynomials α, β and γ such that

$$\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}} = \alpha\Pi_{\frac{d+2}{2}} + \beta\Pi_{\frac{d+4}{2}} + \gamma\mathcal{M}\Pi_{\frac{d}{2}}$$

and no manifestation of the definition relation $\Pi_{\frac{d+4}{2}} = \mathcal{Y}\Pi_{\frac{d+4}{2}} + \mathcal{W}\Pi_{\frac{d}{2}}$ can be subtracted from the presentation. Then $\gamma\mathcal{M}\Pi_{\frac{d}{2}} \in \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}\rangle$, and since $\mathcal{M}$ is of odd degree $\gamma = \mathcal{M}\gamma'$. In the first case the inclusion $A_{j+1} \subseteq B_{d+1}$ follows directly. Consider $\gamma = \mathcal{M}\gamma'$. Since $(\mathcal{Y}+\mathcal{X})\mathcal{W}\Pi_i = \mathcal{Y}\mathcal{W}\Pi_i$ for every $i > 0$, we have that

$$\begin{aligned}\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}} \;&=\; \alpha\Pi_{\frac{d+2}{2}} + \beta\Pi_{\frac{d+4}{2}} + \gamma'\mathcal{M}^2\Pi_{\frac{d}{2}} = \alpha\Pi_{\frac{d+2}{2}} + \beta\Pi_{\frac{d+4}{2}} + \gamma'\mathcal{Y}\mathcal{W}\Pi_{\frac{d}{2}}\\ &=\; \alpha\Pi_{\frac{d+2}{2}} + \beta\Pi_{\frac{d+4}{2}} + \gamma'\mathcal{Y}(\mathcal{Y}\Pi_{\frac{d}{2}+1} + \Pi_{\frac{d}{2}+2}) \in \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}\rangle = B_{d+1}.\end{aligned}$$

Thus $A_{j+1} \subseteq B_{d+1}$.

$\square$

Acknowledgements

We are grateful to Jon Carlson and to Carsten Schultz for useful comments and insightful observations.

Some of this work was done in the framework of the MSRI program "Computational Applications of Algebraic Topology" in the fall semester 2006.

2 Configuration space/Test map scheme

The Configuration Space/Test Map (CS/TM) paradigm (formalized by Živaljević in [27], and also beautifully exposited by Matoušek in [23]) has been very powerful in the systematic derivation of topological lower bounds for problems of Combinatorics and of Discrete Geometry.

In many instances, the problem suggests natural configuration spaces X, Y, a finite symmetry group G, and a test set $Y_0 \subset Y$, where one would try to show that every G-equivariant map $f : X \to Y$ must hit Y_0. The canonical tool is then Dold's theorem, which says that if the group actions are free, then the map f must hit the test set $Y_0 \subset Y$ if the connectivity of X is higher than the dimension of $Y \setminus Y_0$.

For the success of this "canonical approach" one crucially needs that a result such as Dold's theorem is applicable. Thus the group action must be free, so one often reduces the group action to a prime order cyclic subgroup of the full symmetry group, and results may follow only in "the prime case," or with more effort and deeper tools in the prime power case. The main example for this is the Topological Tverberg Problem, which is still not resolved for (d, q) if $d > 1$ and q is not a prime power [23, Section 6.5, page 151]. So in general one has to work much harder when the "canonical" approach fails.

In the following, we present configuration spaces and test maps for the mass partition problem.

2.1 Configuration space

The space of all oriented affine hyperplanes in $\mathbb{R}^d$ can be identified with the sphere S^d. Let $\mathbb{R}^d$ be embedded in $\mathbb{R}^{d+1}$ by $(x_1, ..., x_d) \longmapsto (x_1, ..., x_d, 1)$. Then every oriented affine hyperplane H in $\mathbb{R}^d$ determines a unique oriented hyperplane $\tilde{H}$ through the origin in $\mathbb{R}^{d+1}$ such that $\tilde{H} \cap \mathbb{R}^d = H$, and conversely if the hyperplane at infinity is included. The oriented hyperplane uniquely determined by the unit vector $v \in S^d$ is denoted by H_v and the assumed orientation is determined by the half-space $H_v^{\pm}$. Then $H_{-v}^{-} = H_v^{+}$. The obvious and classically used candidate for the configuration space associated with the problem of testing admissibility of (d, j, k) is

$$Y_{d,k} = \left(S^d\right)^k .$$

The relevant group acting on this space is the Weyl group $W_k = (\mathbb{Z}_2)^k \rtimes S_k$. Each $\mathbb{Z}_2 = (\{+1, -1\}, \cdot)$ acts antipodally on the appropriate copy of S^d (changing the orientation of hyperplane), while S_k acts by permuting copies. The second configuration space that we can use is

$$X_{d,k} = \underbrace{S^d * ... * S^d}_{k \text{ copies}} \;\cong\; S^{dk+k-1}.$$

The elements of $X_{d,k}$ are denoted by $t_1 v_1 + ... + t_k v_k$, with $t_i \geq 0$, $\sum t_1 = 1$, $v_i \in S^d$. The Weyl group W_k acts on $X_{d,k}$ by

$$\begin{aligned} \varepsilon_i \cdot (t_1 v_1 + ... + t_i v_i + ... + t_k v_k) &= t_1 v_1 + ... + t_i(-v_i) + ... + t_k v_k, \\ \pi \cdot (t_1 v_1 + ... + t_i v_i + ... + t_k v_k) &= t_{\pi^{-1}(1)} v_{\pi^{-1}(1)} + ... + t_{\pi^{-1}(i)} v_{\pi^{-1}(i)} + ... + t_{\pi^{-1}(k)} v_{\pi^{-1}(k)}, \end{aligned}$$

where ε_i is the generator of the i-th copy of $\mathbb{Z}_2$ and $\pi \in S_k$ is an arbitrary permutation.

2.2 Test map

Let $\mathcal{M} = \{\mu_1, ..., \mu_j\}$ be a collection of mass distributions in $\mathbb{R}^d$. Let the coordinates of $\mathbb{R}^{2^k}$ be indexed by the elements of the group $(\mathbb{Z}_2)^k$. The Weyl group W_k acts on $\mathbb{R}^{2^k}$ by acting on its coordinate index set $(\mathbb{Z}_2)^k$ in the following way:

$$((\beta_1, ..., \beta_k) \rtimes \pi) \cdot (\alpha_1, .., \alpha_k) = \left(\beta_1 \alpha_{\pi^{-1}(1)}, ..., \beta_k \alpha_{\pi^{-1}(k)}\right).$$

The test map $\phi : Y_{d,k} \to (\mathbb{R}^{2^k})^j$ used with the configuration space $Y_{d,k}$ is a W_k-equivariant map given by

$$\phi\,(v_1, ..., v_k) = \Big(\big(\mu_i(H_{v_1}^{\alpha_1} \cap ... \cap H_{v_k}^{\alpha_k}) - \tfrac{1}{2^k}\mu_i(\mathbb{R}^d)\big)_{(\alpha_1,...,\alpha_k) \in (\mathbb{Z}_2)^k}\Big)_{i \in \{1,..,j\}}.$$

Denote the i-th component of ϕ by ϕ_i, $i = 1, ..., j$.

To define a test map associated with the configuration space $X_{d,k}$, we discuss the $(\mathbb{Z}_2)^k$- and W_k-module structures on $\mathbb{R}^{2^k}$.

All irreducible representations of the group $(\mathbb{Z}_2)^k$ are 1-dimensional. They are in bijection with the homomorphisms (characters) $\chi : (\mathbb{Z}_2)^k \to \mathbb{Z}_2$. These homomorphisms are completely determined by the values on generators ε_1,...,ε_k of $(\mathbb{Z}_2)^k$, i.e. by the vector $(\chi(\varepsilon_1), ..., \chi(\varepsilon_k))$. For $(\alpha_1, .., \alpha_k) \in (\mathbb{Z}_2)^k$ let $V_{\alpha_1...\alpha_k} = \text{span}\{v_{\alpha_1...\alpha_k}\} \subset \mathbb{R}^{2^k}$ denote the 1-dimensional representation given by

$$\varepsilon_i \cdot v_{\alpha_1...\alpha_k} = \alpha_i \ v_{\alpha_1...\alpha_k}$$

The vector $v_{\alpha_1..\alpha_k} \in \{+1, -1\}^{2^k}$ is uniquely determined up to a scalar multiplication by -1. Note that

$$\langle v_{\alpha_1..\alpha_k}, v_{\beta_1..\beta_k} \rangle = 0$$

for $\alpha_1...\alpha_k \neq \beta_1...\beta_k$. For $k = 2$, with the abbreviation $+$ for $+1$, $-$ for -1, the coordinate index set for $\mathbb{R}^4$ is $\{++, +-, -+, --\}$. Then

$$\begin{array}{lcl} v_{++} = (1, 1, 1, 1) & , & v_{+-} = (1, -1, 1, -1), \\ v_{-+} = (1, 1, -1, -1) & , & v_{--} = (1, -1, -1, 1). \end{array}$$

The following decomposition of $(\mathbb{Z}_2)^k$-modules holds, with the index identification $(\mathbb{Z}_2)^k = \{+, -\}^k$,

$$\mathbb{R}^{2^k} \cong V_{+...+} \oplus \sum_{\alpha_1...\alpha_k \in (\mathbb{Z}_2)^k \backslash \{+...+\}} V_{\alpha_1...\alpha_k}$$

where $V_{+...+}$ is the trivial $(\mathbb{Z}_2)^k$-representation. Let R_{2^k} denote the orthogonal complement of $V_{+...+}$ and $\pi : \mathbb{R}^{2^k} \to R_{2^k}$ associated (equivariant) projection. Explicitly

$$R_{2^k} = \{(x_1, .., x_{2^k}) \in \mathbb{R}^{2^k} \mid \sum x_i = 0\} = \sum_{\alpha_1...\alpha_k \in (\mathbb{Z}_2)^k \backslash \{+...+)\}} V_{\alpha_1...\alpha_k}, \tag{5}$$

and

$$\mathbf{x} = (x_1, .., x_{2^k}) \xmapsto{\pi} \tfrac{1}{2^{k-1}} \left(\langle \mathbf{x}, v_{\alpha_1...\alpha_k} \rangle \right)_{\alpha_1...\alpha_k \in (\mathbb{Z}_2)^k \backslash \{+...+\}},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^{2^k}$. Observe that

$$\text{im}\phi = \phi(Y_{d,k}) \subseteq (R_{2^k})^j .$$

Let $\alpha_1...\alpha_k \in (\mathbb{Z}_2)^k$ and let $\eta(\alpha_1...\alpha_k) = \frac{1}{2}(k - \sum \alpha_i)$. The following decomposition of W_k-modules holds

$$\mathbb{R}^{2^k} \cong V_{+...+} \oplus \sum_{n=1}^{k} \sum_{n=\eta(\alpha_1,..,\alpha_k)} V_{\alpha_1...\alpha_k} \cong V_{+...+} \oplus R_{2^k}. \tag{6}$$

The test map $\tau : X_{d,k} \to U_k \times (R_{2^k})^j$ is defined by

$$\begin{array}{rl} \tau\left(t_1 v_1 + ... + t_k v_k\right) = & \left(t_1 - \frac{1}{k}, ..., t_k - \frac{1}{k}\right) \times \\ & \left(\left(t_1^{\frac{1-\alpha_1}{2}} ... t_k^{\frac{1-\alpha_k}{2}} \langle \phi_i\left(v_1, ..., v_k\right), v_{\alpha_1...\alpha_k} \rangle \right)_{\alpha_1...\alpha_k \in (\mathbb{Z}_2)^k \backslash \{+...+\}} \right)_{i=1}^{j} . \end{array}$$

Here $U_k = \{(\xi_1, .., \xi_k) \in \mathbb{R}^k \mid \sum \xi_i = 0\}$ is a W_k-module with an action given by

$$((\beta_1, .., \beta_k) \rtimes \pi) \cdot (\xi_1, .., \xi_k) := \left(\xi_{\pi^{-1}(1)}, .., \xi_{\pi^{-1}(k)}\right).$$

The subgroup $(\mathbb{Z}_2)^k$ acts trivially on U_k. The action on $U_k \times (R_{2^k})^j$ is assumed to be the diagonal action. The test map τ is well defined, continuous and W_k-equivariant.

Example 2.1. The test map $\tau : X_{d,k} \to U_k \times (R_{2^k})^j$ is in the case of $k = 2$ hyperplanes and $j = 1$ measures given by $\tau : X_{d,2} \to U_2 \times R_4 = U_2 \times ((V_{+-} \oplus V_{-+}) \oplus V_{--})$ and

$$\begin{array}{rl} \tau\left(t_1 v_1 + t_2 v_2\right) = & \left(t_1 - \frac{1}{2}, t_2 - \frac{1}{2},\right. \\ & \left. t_1 \langle \phi\left(v_1, v_2\right), v_{-+} \rangle, t_2 \langle \phi\left(v_1, v_2\right), v_{+-} \rangle, t_1 t_2 \langle \phi\left(v_1, v_2\right), v_{--} \rangle \right) \end{array}$$

where

$$\phi\left(v_1, v_2\right) = \left(\mu_i(H_{v_1}^{\alpha_1} \cap H_{v_2}^{\alpha_2}) - \tfrac{1}{4}\mu(\mathbb{R}^d)\right)_{\alpha_1 \alpha_2 \in (\mathbb{Z}_2)^2} \in \mathbb{R}^4.$$

2.3 The test space

The test spaces for the maps ϕ and τ are the origins of $(R_{2^k})^j$ and $U_k \times (R_{2^k})^j$, respectively. The constructions that we performed in this section satisfy the usual hypotheses for the CS/TM scheme.

Proposition 2.2.

(i) *For a collection of mass distributions* $\mathcal{M} = \{\mu_1, .., \mu_j\}$ *let* $\phi : Y_{d,k} \to (R_{2^k})^j$ *and* $\tau : X_{d,k} \to U_k \times (R_{2^k})^j$ *be the corresponding test maps. If*

$$(0, ..., 0) \in \phi\,(Y_{d,k}) \qquad \text{or} \qquad (0, ..., 0) \in \tau\,(X_{d,k})$$

then there exists an arrangement of k *hyperplanes* $\mathcal{H}$ *in* $\mathbb{R}^d$ *equiparting the collection* $\mathcal{M}$*.*

(ii) *If there is no* W_k*-equivariant map with respect to the actions defined above,*

$$\begin{array}{llll} Y_{d,k} \to (R_{2^k})^j \setminus \{(0, ..., 0)\}, & \text{or} & Y_{d,k} \to S\left((R_{2^k})^j\right) \approx S^{j(2^k-1)-1}, & \text{or} \\ X_{d,k} \to U_k \times (R_{2^k})^j \setminus \{(0, ..., 0)\}, & \text{or} & X_{d,k} \to S\left(U_k \times (R_{2^k})^j\right) \approx S^{j(2^k-1)+k-2}, & \end{array}$$

then the triple (d, j, k) *is admissible.*

(iii) *Specifically, for* $k = 2$*, if there is no* $D_8 \cong W_2$ *equivariant map, with the already defined actions,*

$$\begin{array}{llll} Y_{d,2} \to (R_4)^j \setminus \{(0, ..., 0)\}, & \text{or} & Y_{d,2} \to S\left((R_4)^j\right) \approx S^{3j-1}, & \text{or} \\ X_{d,2} \to U_2 \times (R_4)^j \setminus \{(0, ..., 0)\}, & \text{or} & S^{2d+1} \approx X_{d,2} \to S\left(U_2 \times (R_4)^j\right) \approx S^{3j}, & \end{array}$$

then the triple $(d, j, 2)$ *is admissible.*

Remark 2.3. The action of W_k on the sphere $S(U_2 \times (R_4)^j)$ is *fixed point free, but not free.* For $k = 2$, the action of the unique $\mathbb{Z}_4$ subgroup of $W_2 = D_8$ on the sphere $S(U_2 \times (R_4)^j)$ is fixed point free.

The necessary condition for the non-existence of an equivariant W_k-map

$$X_{d,k} \to S(U_k \times (R_{2^k})^j)$$

implied by the equivariant Kuratowski–Dugundji theorem [3, Theorem 1.3, page 25] is

$$dk + k - 1 > j(2^k - 1) + k - 2 \iff d \geq \tfrac{2^k-1}{k} j \ . \tag{7}$$

For $k = 2$ the condition (7) becomes

$$d \geq \lceil \tfrac{3}{2} j \rceil. \tag{8}$$

3 The Fadell–Husseini index theory

3.1 Equivariant cohomology

Let X be a G -space and $X \to \mathrm{E}G \times_G X \overset{\pi_X}{\to} \mathrm{B}G$ the associated universal bundle, with X as a typical fibre. $\mathrm{E}G$ is a contractible cellular space on which G acts freely, and $\mathrm{B}G := \mathrm{E}G/G$. The space $\mathrm{E}G \times_G X = (\mathrm{E}G \times X)/G$ is called the *Borel construction* of X in respect to the action of G. The *equivariant cohomology* of X is the ordinary cohomology of the Borel construction $\mathrm{E}G \times_G X$,

$$H^*_G(X) := H^*(\mathrm{E}G \times_G X).$$

The equivariant cohomology is a module over the ring $H^*_G(pt) = H^*(\mathrm{B}G)$. When X is a free G-space the homotopy equivalence $\mathrm{E}G \times_G X \simeq X/G$ induces a natural isomorphism

$$H^*_G(X) \cong H^*(X/G).$$

The universal bundle $X \to \mathrm{E}G \times_G X \stackrel{\pi_X}{\to} \mathrm{B}G$, for coefficients in the ring R, induces a Serre spectral sequence converging to the graded group $\mathrm{Gr}(H^*_G(X,R))$ associated with $H^*_G(X,R)$ appropiately filtered. In this paper "ring" means commutative ring with the unit element. The E_2-term is given by

$$E_2^{p,q} \cong H^p(\mathrm{B}G, \mathcal{H}^q(X,R)), \tag{9}$$

where $\mathcal{H}^q(X,R)$ is a system of local coefficients. For a discrete group G, the E_2-term of the spectral sequence can be interpreted as the cohomology of the group G with coefficients in the G-module $H^*(X,R)$,

$$E_2^{p,q} \cong H^p(G, H^q(X,R)). \tag{10}$$

3.2 $\mathrm{Index}_{G.R}$ and $\mathrm{Index}^k_{G,R}$

Let X be a G-space, R a ring and π^*_X the ring homomorphism in cohomology

$$\pi^*_X : H^*(\mathrm{B}G, R) \to H^*(\mathrm{E}G \times_G X, R)$$

induced by mappings

$$\begin{array}{ccc} X & & \mathrm{E}G \times_G X \\ \downarrow & & \downarrow \\ \{p\} & & \mathrm{E}G \times_G \{p\} \quad \approx \mathrm{B}G. \end{array}$$

The *Fadell–Husseini (index-valued) index* of a G-space X is the kernel ideal of π^*_X,

$$\mathrm{Index}_{G,R} X := \ker \pi^*_X \subseteq H^*(\mathrm{B}G, R).$$

The Serre spectral sequence (9) yields a representation of the homomorphism π^*_X as the composition

$$H^*(\mathrm{B}G, R) \to E_2^{*,0} \to E_3^{*,0} \to E_4^{*,0} \to \dots \to E_\infty^{*,0} \subseteq H^*(\mathrm{E}G \times_G X, R).$$

The *k-th Fadell–Husseini index* is defined by

$$\begin{array}{rcll} \mathrm{Index}^k_{G,R} X & = & \ker\left(H^*(\mathrm{B}G,R) \to E_k^{*,0}\right), & \qquad k \geq 2, \\ \mathrm{Index}^1_{G,R} X & = & \{0\}. & \end{array}$$

From the definitions the following properties of indexes can be derived.

Proposition 3.1. *Let X, Y be G-spaces and $f : X \to Y$ a G-map.*
(1) $\mathrm{Index}^k_{G,R} X \subseteq H^*(\mathrm{B}G, R)$ *is an ideal, for every $k \in \mathbb{N}$;*
(2) $\mathrm{Index}^1_{G,R} X \subseteq \mathrm{Index}^2_{G,R} X \subseteq \mathrm{Index}^3_{G,R} X \subseteq \dots \subseteq \mathrm{Index}_{G,R} X$
(3) $\bigcup_{k \in \mathbb{N}} \mathrm{Index}^k_{G,R} X = \mathrm{Index}_{G,R} X$

Proposition 3.2. *Let X and Y be G-spaces and $f : X \to Y$ a G-map. Then*

$$\mathrm{Index}_{G,R}(X) \supseteq \mathrm{Index}_{G,R}(Y)$$

and for every $k \in \mathbb{N}$

$$\mathrm{Index}^k_{G,R}(X) \supseteq \mathrm{Index}^k_{G,R}(Y).$$

Proof. Functoriality of all constructions implies that the following diagrams commute

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \{\mathrm{pt}\} & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{E}G \times_G X \arrow[rr, "\hat{f}"] \arrow[dr, "\pi_X"'] & & \mathrm{E}G \times_G Y \arrow[dl, "\pi_Y"] \\ & \mathrm{B}G & \end{tikzcd} \qquad \begin{tikzcd} H^*(\mathrm{E}G \times_G X, R) & & H^*(\mathrm{E}G \times_G Y, R) \arrow[ll, "f^*"'] \\ & H^*(\mathrm{B}G, R) \arrow[ul, "\pi^*_X"] \arrow[ur, "\pi^*_Y"'] & \end{tikzcd}$$

$\pi_X = \hat{f} \circ \pi_Y$ and $\pi^*_X = \pi^*_Y \circ f^*$. Thus $\ker \pi^*_X \supseteq \ker \pi^*_Y$. $\square$

Example 3.3. S^n is a $\mathbb{Z}_2$-space with the antipodal action. The action is free and therefore

$$\mathrm{E}\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^n \simeq S^n/\mathbb{Z}_2 \approx \mathbb{R}\mathrm{P}^n \;\Rightarrow\; H^*_{\mathbb{Z}_2}(S^n, R) \cong H^*(\mathbb{R}\mathrm{P}^n, R).$$

1. $R = \mathbb{F}_2$: The cohomology ring $H^*(\mathrm{B}\mathbb{Z}_2, \mathbb{F}_2) = H^*(\mathbb{R}\mathrm{P}^\infty, \mathbb{F}_2)$ is the polynomial ring $\mathbb{F}_2[t]$ where $\deg(t) = 1$. The $\mathbb{Z}_2$-index of S^n is the principal ideal generated by t^{n+1}:

$$\mathrm{Index}_{\mathbb{Z}_2,\mathbb{F}_2} S^n = \mathrm{Index}^{n+2}_{\mathbb{Z}_2,\mathbb{F}_2} S^n = \langle t^{n+1} \rangle \subseteq \mathbb{F}_2[t].$$

2. $R = \mathbb{Z}$: The cohomology ring $H^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}) = H^*(\mathbb{R}\mathrm{P}^\infty, \mathbb{Z})$ is the quotient polynomial ring $\mathbb{Z}[\tau]/\langle 2\tau \rangle$ where $\deg(\tau) = 2$. The $\mathbb{Z}_2$-index of S^n is the principal ideal

$$\mathrm{Index}_{\mathbb{Z}_2,\mathbb{Z}} S^n = \mathrm{Index}^{n+2}_{\mathbb{Z}_2,\mathbb{Z}} S^n = \begin{cases} \langle \tau^{\frac{n+1}{2}} \rangle, & \text{for } n \text{ odd} \\ \langle \tau^{\frac{n+2}{2}} \rangle, & \text{for } n \text{ even} \end{cases}.$$

Example 3.4. Let G be a finite group and H a subgroup of index 2. Then $H \lhd G$ and $G/H \cong \mathbb{Z}_2$. Let V be the 1-dimensional real representation of G defined for $v \in V$ by

$$g \cdot v = \begin{cases} v, & \text{for } g \in H \\ -v, & \text{for } g \notin H \end{cases}.$$

There is a G-homeomorphism $S(V) \approx \mathbb{Z}_2$. Therefore by [16, last equation on page 34]:

$$\mathrm{E}G \times_G S(V) \approx \mathrm{E}G \times_G (G/H) \approx (\mathrm{E}G \times_G G)\,/H \approx \mathrm{E}G/H \approx \mathrm{B}H$$

and

$$\mathrm{Index}_{G,R} S(V) = \ker\left(\mathrm{res}^G_H : H^*(G,R) \to H^*(H,R)\right). \tag{11}$$

3.3 The restriction map and the index

Let X be a G-space and K a subgroup. Then there is a commutative diagram of fibrations [10, pages 179-180]:

$$\begin{array}{ccc} \mathrm{E}G \times_G X & \xleftarrow{f} & \mathrm{E}G \times_K X \\ \downarrow & & \downarrow \\ \mathrm{B}G = \mathrm{E}G/G & \xleftarrow{\mathrm{B}i} & \mathrm{E}G/K = \mathrm{B}K \end{array} \tag{12}$$

induced by inclusion $i : K \subset G$. Here $\mathrm{E}G$ in the lower right corner is understood as a K-space and consequently a model for $\mathrm{E}K$. The map $\mathrm{B}i$ is a map between classifying spaces induced by inclusion i. Now with coefficients in the ring R we define

$$\mathrm{res}^G_K := H^*(f) : H^*(\mathrm{E}G \times_G X, R) \to H^*(\mathrm{E}G \times_K X, R).$$

If G is a finite group, then the induced map on the cohomology of the classifying spaces

$$\mathrm{res}^G_K = (\mathrm{B}i)^* : H^*(\mathrm{B}G, R) \to H^*(\mathrm{B}K, R)$$

coincides with the restriction homomorphism between group cohomologies

$$\mathrm{res}^G_K : H^*(G,R) \to H^*(K,R).$$

Proposition 3.5. *Let X be a G-space, and K and L subgroups of G.*
(A) *The morphism of fibrations* (12) *provides the following commutative diagram in cohomology*

$$\begin{array}{ccc} H^*(\mathrm{E}G \times_G X, R) & \xrightarrow{\mathrm{res}^G_K} & H^*(\mathrm{E}G \times_K X, R) \\ \uparrow{\scriptstyle \pi_X^*} & & \uparrow{\scriptstyle \pi_X^*} \\ H^*(\mathrm{B}G, R) & \xrightarrow{\mathrm{res}^G_K} & H^*(\mathrm{B}K, R) \end{array} \tag{13}$$

(B) *For every $x \in H^*(\mathrm{B}G, R)$ and $y \in H^*(\mathrm{E}G \times_G X, R)$,*

$$\mathrm{res}^G_K(x \cdot y) = \mathrm{res}^G_K(x) \cdot \mathrm{res}^G_K(y).$$

(C) $L \subset K \subset G \;\Rightarrow\; \mathrm{res}^G_L = \mathrm{res}^K_L \circ \mathrm{res}^G_K$.

(D) *The map of fibrations* (12) *induces a morphism of Serre spectral sequences*

$$\Gamma^{*,*}_i : E^{*,*}_i(\mathrm{E}G \times_G X, R) \to E^{*,*}_i(\mathrm{E}K \times_K X, R)$$

such that
(1) $\Gamma^{*,*}_\infty = \mathrm{res}^G_K : H^{*+*}(\mathrm{E}G \times_G X, R) \to H^{*+*}(\mathrm{E}G \times_K X, R)$,
(2) $\Gamma^{*,0}_2 = \mathrm{res}^G_K : H^*(\mathrm{B}G, R) \to H^*(\mathrm{B}K, R)$.

(E) *Let R and S be commutative rings and $\phi : R \to S$ a ring homomorphism. There are morphisms:*
(1) *in equivariant cohomology* $\Phi^* : H^*(\mathrm{E}G \times_G X, R) \to H^*(\mathrm{E}G \times_G X, S)$,
(2) *in group cohomology* $\Phi^* : H^*(G, R) \to H^*(G, S)$, *and*
(3) *between Serre spectral sequences* $\Phi^{*,*}_i : E^{*,*}_i(\mathrm{E}G \times_G X, R) \to E^{*,*}_i(\mathrm{E}G \times_G X, S)$,
induced by ϕ such that the diagram on Figure 1 commutes.

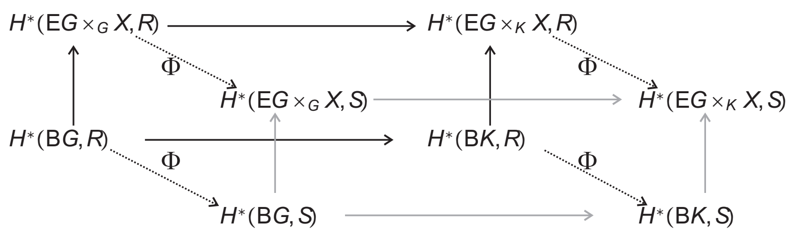

Figure 1: Diagram induced by coefficient map $\phi : R \to S$

Remark 3.6. By the morphism of spectral sequences in properties (D) and (E) we mean that

$$\Gamma^{*,*}_i \circ \partial_i = \partial_i \circ \Gamma^{*,*}_i \qquad \text{and} \qquad \Phi^{*,*}_i \circ \partial_i = \partial_i \circ \Phi^{*,*}_i.$$

These relations are applied in the situations where the right hand side is $\neq 0$ for a particular element x, to imply that the left hand side $\Gamma^{*,*}_i \circ \partial_i(x)$ or $\Phi^{*,*}_i \circ \partial_i(x)$ is also $\neq 0$. In particular, then $\partial_i(x) \neq 0$.

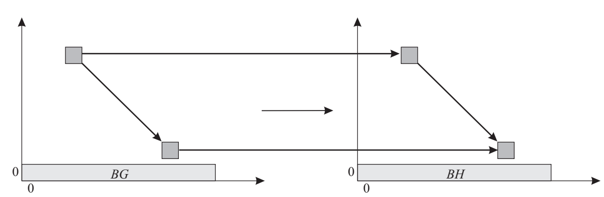

Figure 2: Illustration of Proposition 3.5 (D) and (E)

Proposition 3.7. *Let X be a G-space and K a subgroup of G. Let R and S be rings and $\phi : R \to S$ a ring homomorphism. Then*
(1) $\mathrm{res}^G_K(\mathrm{Index}_{G,R}X) \subseteq \mathrm{Index}_{K,R}X$,
(2) $\mathrm{res}^G_K\left(\mathrm{Index}^r_{G,R}X\right) \subseteq \mathrm{Index}^r_{K,R}X$ *for every $r \in \mathbb{N}$,*
(3) $\Phi^*(\mathrm{Index}_{G,R}X) \subseteq \mathrm{Index}_{G,S}X$,
(4) $\Phi^*(\mathrm{Index}^r_{G,R}X) \subseteq \mathrm{Index}^r_{G,S}X$.

Proof. The assertion about the $\mathrm{Index}_{G,R}$ follows from the diagram (13) and Figure 1. The commutative diagrams

$$\begin{array}{ccc} E^{*,0}_r(\mathrm{E}G \times_G X) & \xrightarrow{\Gamma^{*,0}_r} & E^{*,0}_r(\mathrm{E}K \times_K X) \\ \uparrow & & \uparrow \\ H^*(\mathrm{B}G) & \xrightarrow{\mathrm{res}^G_K} & H^*(\mathrm{B}K) \end{array}$$

and

$$\begin{array}{ccc} E_r^{*,0}(\mathrm{E}G \times_G X, R) & \xrightarrow{\Phi_r^{*,0}} & E_r^{*,0}(\mathrm{E}G \times_G X, S) \\ \uparrow & & \uparrow \\ H^*(\mathrm{B}G, R) & \xrightarrow{\Phi^*} & H^*(\mathrm{B}G, S) \end{array}$$

imply the partial index $\mathrm{Index}^r_{G,R}$ assertions. □

3.4 Basic calculations of the index

3.4.1 The index of a product

Let X be a G-space and Y an H-space. Then $X \times Y$ has the natural structure of a $G \times H$ space. What is the relation between three indexes $\mathrm{Index}_{G\times H}(X \times Y)$, $\mathrm{Index}_G(X)$, and $\mathrm{Index}_H(Y)$? Using the Künneth formula one can prove the following proposition [12, Corollary 3.2], [28, Proposition 2.7] when the coefficient ring is a field.

Proposition 3.8. *Let X be a G-space and Y an H-space and*

$$H^*(BG, \Bbbk) \cong \Bbbk[x_1, ..., x_n], \qquad\qquad H^*(BH, \Bbbk) \cong \Bbbk[y_1, ..., y_n]$$

the cohomology rings of the associated classifying spaces with coefficients in the field $\Bbbk$. If

$$\mathrm{Index}_{G,\Bbbk} X = \langle f_1, ..., f_i\rangle \qquad \textit{and} \qquad \mathrm{Index}_{H,\Bbbk}(Y) = \langle g_1, ..., g_j\rangle,$$

then

$$\mathrm{Index}_{G\times H,\Bbbk} X = \langle f_1, ..., f_i, g_1, ..., g_j\rangle \subseteq \Bbbk[x_1, ..., x_n, y_1, ..., y_n].$$

The $(\mathbb{Z}_2)^k$-index of a sphere product can be computed using this proposition and Example 3.3.

Corollary 3.9. *Let $S^{n_1} \times ... \times S^{n_k}$ be a $(\mathbb{Z}_2)^k$-space with the product action. Then*

$$\mathrm{Index}_{(\mathbb{Z}_2)^k,\mathbb{F}_2} S^{n_1} \times ... \times S^{n_k} = \langle t_1^{n_1+1}, ..., t_k^{n_k+1}\rangle \subseteq \mathbb{F}_2[t_1, ..., t_k].$$

Unfortunately when the coefficient ring is not a field the claim of Proposition 3.8 does not hold.

Example 3.10. Let $S^n \times S^n$ be a $(\mathbb{Z}_2)^2$-space with the product action. From the previous corollary

$$\mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{F}_2} S^n \times S^n = \langle t_1^{n+1}, t_2^{n+1}\rangle \subseteq \mathbb{F}_2[t_1, t_2] = H^*((\mathbb{Z}_2)^2, \mathbb{F}_2). \tag{14}$$

To determine $\mathbb{Z}$-index we proceed in two steps.
Cohomology ring $H^*((\mathbb{Z}_2)^2, \mathbb{Z})$: Following [19, Section 4.1, page 508] the short exact sequence of coefficients

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{j} \mathbb{F}_2 \to 0 \tag{15}$$

induces a long exact sequence in group cohomology [6, Proposition 6.1, page 71] which in this case reduces to a sequence of short exact sequences for $k > 0$,

$$0 \to H^k((\mathbb{Z}_2)^2, \mathbb{Z}) \xrightarrow{j^*} H^k((\mathbb{Z}_2)^2, \mathbb{F}_2) \to H^{k+1}((\mathbb{Z}_2)^2, \mathbb{Z}) \to 0. \tag{16}$$

Therefore, like in [19, Proposition 4.1, page 508],

$$H^*((\mathbb{Z}_2)^2, \mathbb{Z}) \cong \mathbb{Z}[\tau_1, \tau_2] \otimes \mathbb{Z}[\mu] \tag{17}$$

where $\deg \tau_1 = \deg \tau_2 = 2$, $\deg \mu = 3$ and

$$2\tau_1 = 2\tau_2 = 2\mu = 0 \qquad \text{and} \qquad \mu^2 = \tau_1\tau_2(\tau_1 + \tau_2).$$

The ring morphism $j : \mathbb{Z} \to \mathbb{F}_2$, from the coefficient exact sequence (15), induces a morphism in group cohomology $j^* : H^*((\mathbb{Z}_2)^2, \mathbb{Z}) \to H^*((\mathbb{Z}_2)^2, \mathbb{F}_2)$ given by:

$$\tau_1 \longmapsto t_1^2,\ \tau_2 \longmapsto t_2^2,\ \mu \longmapsto t_1t_2(t_1 + t_2). \tag{18}$$

The arguments used in computation of the cohomology with integer coefficients come from the Bockstein spectral sequence [5], [8, pages 104-110] associated with the exact couple

$$\begin{tikzcd} H^*((\mathbb{Z}_2)^2,\mathbb{Z}) \arrow[rr, "p"] & & H^*((\mathbb{Z}_2)^2,\mathbb{Z}) \arrow[dl, "q"] \\ & H^*((\mathbb{Z}_2)^2,\mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}$$

where $\deg(p)=\deg(q)=0$ and $\deg(\delta)=1$. The first differential $d_1=q\circ\delta$ coincides with the the first Steenrod square $\mathrm{Sq}^1 : H^*((\mathbb{Z}_2)^2,\mathbb{F}_2)\to H^{*+1}((\mathbb{Z}_2)^2,\mathbb{F}_2)$ and therefore is given by

$$1\mapsto 0,\ t_1\mapsto t_1^2,\ t_2\mapsto t_2^2.$$

Consequently, $t_1t_2\mapsto t_1^2t_2+t_1t_2^2$. The spectral sequence stabilizes in the second step since the derived couple is

$$\begin{tikzcd} 0 \arrow[rr] & & 0 \arrow[dl, "q"] \\ & \mathbb{F}_2 \arrow[ul, "\delta"] & \end{tikzcd}$$

where $\mathbb{F}_2$ is in dimension 0.
$\mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{Z}}S^n\times S^n$: The $(\mathbb{Z}_2)^2$ action on $S^n\times S^n$, as product of antipodal actions, is free and therefore

$$\mathrm{E}(\mathbb{Z}_2)^2\times_{(\mathbb{Z}_2)^2}(S^n\times S^n)\ \simeq\ (S^n\times S^n)/(\mathbb{Z}_2)^2\approx \mathbb{R}P^n\times\mathbb{R}P^n.$$

Using equality (14), Proposition 3.5.E.3 on the coefficient morphism $j:\mathbb{Z}\to\mathbb{F}_2$, the isomorphism

$$H^*_{(\mathbb{Z}_2)^2}(S^n\times S^n,\mathbb{Z})\cong H^*(\mathbb{R}P^n\times\mathbb{R}P^n,\mathbb{Z})$$

and the existence of the $(\mathbb{Z}_2)^2$-inclusions

$$S^{n-1}\times S^{n-1}\subset S^n\times S^n\subset S^{n+1}\times S^{n+1}$$

it can be concluded that

$$\mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{Z}}S^n\times S^n=\begin{cases}\langle \tau_1^{\frac{n+1}{2}},\tau_2^{\frac{n+1}{2}}\rangle, & \text{for } n \text{ odd}\\ \langle \tau_1^{\frac{n+2}{2}},\tau_2^{\frac{n+2}{2}},\tau_1^{\frac{n}{2}}\mu,\tau_2^{\frac{n}{2}}\mu\rangle, & \text{for } n \text{ even}\end{cases}\subseteq H^*((\mathbb{Z}_2)^2,\mathbb{Z}). \tag{19}$$

3.4.2 The index of a sphere

We need to know how to compute the index of a sphere that is not equipped by the antipodal $\mathbb{Z}_2$-action only. The following three propositions will be of some help [12, Proposition 3.13], [28, Proposition 2.9].

Proposition 3.11. *Let G be a finite group and V an n-dimensional complex representation of G. Then*

$$\mathrm{Index}_{G,\mathbb{Z}}S(V)=\langle \mathrm{c}_n(V)\rangle\subset H^*(G,\mathbb{Z})$$

where $\mathrm{c}_n(V)$ is the n-th Chern class of the bundle $V\to\mathrm{E}G\times_G V\to\mathrm{B}G$.

Proof. If the group G acts on $H^*(S(V),\mathbb{Z})$ trivially, then from the Serre spectral sequence of the sphere bundle

$$S(V)\to\mathrm{E}G\times_G S(V)\to\mathrm{B}G$$

it follows that

$$\mathrm{Index}_{G,\mathbb{Z}}S(V)=\langle \mathrm{e}(V_G)\rangle\subset H^*(G,\mathbb{Z}),$$

where $\mathrm{e}(V)$ is the Euler class of the bundle $V\to\mathrm{E}G\times_G V\to\mathrm{B}G$. Now V is a complex G-representation, therefore the group G acts trivially on $H^*(S(V),\mathbb{Z})$. From [17, Exercise 3, page 261] it follows that

$$\mathrm{e}(V_G)=\mathrm{c}_n(V_G)$$

and the statement is proved. ∎

Proposition 3.12. *Let U, V be two G-representations and let $S(U)$, $S(V)$ be the associated G-spheres. Let R be a ring and assume that $H^*(S(U),R)$, $H^*(S(V),R)$ are trivial G-modules. If* $\mathrm{Index}_{G,R}(S(U)) = \langle f\rangle \subseteq H^*(\mathrm{B}G,R)$ *and* $\mathrm{Index}_{G,R}(S(V)) = \langle g\rangle \subseteq H^*(\mathrm{B}G,R)$, *then*

$$\mathrm{Index}_{G,R}S(U\oplus V) = \langle f\cdot g\rangle \subseteq H^*(\mathrm{B}G,R).$$

Proposition 3.13. (A) *Let V be the 1-dimensional $(\mathbb{Z}_2)^k$-representation with the associated ± 1 vector $(\alpha_1,..,\alpha_k)\in(\mathbb{Z}_2)^k$ (as defined in Section 2). Then*

$$\mathrm{Index}_{(\mathbb{Z}_2)^k,\mathbb{F}_2}S(V) = \langle \bar\alpha_1 t_1 + ... + \bar\alpha_k t_k\rangle \subseteq \mathbb{F}_2[t_1,...,t_k],$$

where $\bar\alpha_i = 0$ if $\alpha_i = 1$, and $\bar\alpha_i = 1$ if $\alpha_i = -1$.

(B) *Let U be an n-dimensional $(\mathbb{Z}_2)^k$-representation with a decomposition $U\cong V_1\oplus ...\oplus V_n$ into 1-dimensional $(\mathbb{Z}_2)^k$-representations $V_1,...,V_n$. If $(\alpha_{1i},..,\alpha_{ki})\in(\mathbb{Z}_2)^k$ is the associated ± 1 vector of V_i, then*

$$\mathrm{Index}_{(\mathbb{Z}_2)^k,\mathbb{F}_2}S(U) = \Big\langle \prod_{i=1}^{n}(\bar\alpha_{1i}t_1 + ... + \bar\alpha_{ki}t_k)\Big\rangle \subseteq \mathbb{F}_2[t_1,...,t_k].$$

Example 3.14. Let V_{-+}, V_{+-} and V_{--} be 1-dimensional real $(\mathbb{Z}_2)^2$-representations introduced in Section 2.2. Then by Proposition 3.13

$$\mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{F}_2}S(V_{-+}) = \langle t_1\rangle,\qquad \mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{F}_2}S(V_{+-}) = \langle t_2\rangle,\qquad \mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{F}_2}S(V_{--}) = \langle t_1+t_2\rangle.$$

On the other hand, Example 3.4 and the restriction diagram (41) imply that

$$\mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{Z}}S(V_{-+}) = \langle \tau_1,\mu\rangle,\quad \mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{Z}}S(V_{+-}) = \langle \tau_2,\mu\rangle,\quad \mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{Z}}S(V_{--}) = \langle \tau_1+\tau_2,\mu\rangle.$$

4 The cohomology of D_8 and the restriction diagram

The dihedral group $W_2 = D_8 = (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2 = (\langle\varepsilon_1\rangle\times\langle\varepsilon_2\rangle)\rtimes\langle\sigma\rangle$ can be presented by

$$D_8 = \langle \varepsilon_1,\sigma \mid \varepsilon_1^2 = \sigma^2 = (\varepsilon_1\sigma)^4 = 1\rangle.$$

Then $\langle\varepsilon_1\sigma\rangle\cong\mathbb{Z}_4$ and $\varepsilon_2 = \sigma\varepsilon_1\sigma$.

4.1 The poset of subgroups of D_8

The poset $\mathrm{Sub}(G)$ denotes the collection of all nontrivial subgroups of a given group G ordered by inclusion. The poset $\mathrm{Sub}(G)$ can be interpreted as a small category $\mathfrak{G}$ in the usual way:

- $\mathrm{Ob}(\mathfrak{G}) = \mathrm{Sub}(G)$,
- for every two objects H and K, subgroups of G, there is a unique morphism $f_{H,K}: H\to K$ if $H\supseteq K$, and no morphism if $H\not\supseteq K$, i.e.

$$\mathrm{Mor}(H,K) = \begin{cases} \{f_{H,K}\}, & H\supseteq K,\\ \emptyset, & H\not\supseteq K,\end{cases}.$$

The Hasse diagram of the poset $\mathrm{Sub}(D_8)$ is presented in the following drawing.

$$\begin{array}{ccccccccc}
 & & & & D_8 & & & & \\
 & & & \swarrow & \downarrow & \searrow & & & \\
 & & \boxed{\begin{array}{c} H_1 \\ \langle\varepsilon_1,\varepsilon_2\rangle \\ \mathbb{Z}_2\times\mathbb{Z}_2\end{array}} & & \boxed{\begin{array}{c} H_2 \\ \langle\varepsilon_1\sigma\rangle \\ \mathbb{Z}_4\end{array}} & & \boxed{\begin{array}{c} H_3 \\ \langle\varepsilon_1\varepsilon_2,\sigma\rangle \\ \mathbb{Z}_2\times\mathbb{Z}_2\end{array}} & & \\
 & \swarrow & \downarrow & \searrow & \downarrow & \swarrow & \downarrow & \searrow & \\
\boxed{\begin{array}{c} K_1 \\ \langle\varepsilon_1\rangle \\ \mathbb{Z}_2\end{array}} & & \boxed{\begin{array}{c} K_2 \\ \langle\varepsilon_2\rangle \\ \mathbb{Z}_2\end{array}} & & \boxed{\begin{array}{c} K_3 \\ \langle\varepsilon_1\varepsilon_2\rangle \\ \mathbb{Z}_2\end{array}} & & \boxed{\begin{array}{c} K_4 \\ \langle\sigma\rangle \\ \mathbb{Z}_2\end{array}} & & \boxed{\begin{array}{c} K_5 \\ \langle\varepsilon_1\varepsilon_2\sigma\rangle \\ \mathbb{Z}_2\end{array}}
\end{array}$$

4.2 The cohomology ring $H^*(D_8, \mathbb{F}_2)$

The dihedral group D_8 is an example of a wreath product. Therefore the associated classifying space can, as in [1, page 117], be written explicitly as

$$\mathrm{B}D_8 = \mathrm{B}(\mathbb{Z}_2)^2 \times_{\mathbb{Z}_2} \mathrm{E}\mathbb{Z}_2 \approx (\mathrm{B}(\mathbb{Z}_2)^2) \times_{\mathbb{Z}_2} \mathrm{E}\mathbb{Z}_2,$$

where $\mathbb{Z}_2 = \langle\sigma\rangle$ acts on $(\mathrm{B}\mathbb{Z}_2)^2$ by interchanging coordinates. Presented in this way $\mathrm{B}D_8$ is the Borel construction of the $\mathbb{Z}_2$-space $(\mathrm{B}\mathbb{Z}_2)^2$. Thus $\mathrm{B}D_8$ fits in a fibration

$$\mathrm{B}(\mathbb{Z}_2)^2 \to (\mathrm{B}(\mathbb{Z}_2)^2) \times_{\mathbb{Z}_2} \mathrm{E}\mathbb{Z}_2 \to \mathrm{B}\mathbb{Z}_2. \tag{20}$$

There is an associated Serre spectral sequence with E_2-term

$$E_2^{p,q} = \begin{cases} H^p(\mathrm{B}\mathbb{Z}_2, \mathcal{H}^q\left(\mathrm{B}(\mathbb{Z}_2)^2, \mathbb{F}_2\right)) \\ H^p(\mathbb{Z}_2, H^q\left((\mathbb{Z}_2)^2, \mathbb{F}_2\right)) \end{cases} \implies \begin{cases} H^{p+q}(\mathrm{B}D_8, \mathbb{F}_2) \\ H^{p+q}(D_8, \mathbb{F}_2) \end{cases} \tag{21}$$

which converges to cohomology of the group D_8 with $\mathbb{F}_2$-coefficients. This spectral sequence is also the Lyndon-Hochschild-Serre (LHS) spectral sequence [1, Section IV.1, page 116] associated with the group extension sequence:

$$1 \to (\mathbb{Z}_2)^2 \to D_8 \to D_8/(\mathbb{Z}_2)^2 \to 1.$$

In [1, Theorem 1.7, page 117] it is proved that the spectral sequence (21) collapses in the E_2-term. Therefore, to compute the cohomology of D_8 we only need to read the E_2-term.

Lemma 4.1.
(i) $H^*\left((\mathbb{Z}_2)^2, \mathbb{F}_2\right) \cong_{\mathrm{ring}} \mathbb{F}_2[a, a+b]$, *where* $\deg(a) = \deg(a+b) = 1$ *and the* $\mathbb{Z}_2$*-action induced by* σ *is given by* $\sigma \cdot a = a + b$.
(ii) $H^*\left((\mathbb{Z}_2)^2, \mathbb{F}_2\right)^{\mathbb{Z}_2} \cong_{\mathrm{ring}} \mathbb{F}_2[b, a(a+b)]$.
(iii) $H^i\left((\mathbb{Z}_2)^2, \mathbb{F}_2\right) \cong_{\mathbb{Z}_2\text{-module}} \mathbb{F}_2[\mathbb{Z}_2]^{s_{i,1}} \oplus \mathbb{F}_2^{s_{i,2}}$, *where* $s_{i,1} \geq 0$, $s_{i,2} \geq 0$ *and* $\mathbb{F}_2[\mathbb{Z}_2]$ *denotes a free* $\mathbb{Z}_2$*-module and* $\mathbb{F}_2$ *a trivial one.*
(iv) $E_2^{*,i} = H^*(\mathbb{Z}_2, H^i\left((\mathbb{Z}_2)^2, \mathbb{F}_2\right)) \cong_{\mathrm{ring}} H^*(\mathbb{Z}_2, \mathbb{F}_2)^{\oplus s_{i,2}} \oplus \mathbb{F}_2^{s_{i,1}}$, *where* $\mathbb{F}_2^{s_{i,1}}$ *denotes a ring concentrated in dimension* 0.

Proof. (i) The statement follows from the observation that $\mathrm{B}(\mathbb{Z}_2)^2 \approx \left(\mathrm{B}(\mathbb{Z}_2)\right)^2$, and consequently

$$H^*\left((\mathbb{Z}_2)^2, \mathbb{F}_2\right) \cong_{\mathrm{ring}} H^*\left(\mathbb{Z}_2, \mathbb{F}_2\right) \otimes H^*\left(\mathbb{Z}_2, \mathbb{F}_2\right) \cong_{\mathrm{ring}} \mathbb{F}_2[a] \otimes \mathbb{F}_2[a+b].$$

The $\mathbb{Z}_2$-action interchanges copies on the left hand side. Generators on the right hand side are chosen such that the $\mathbb{Z}_2$-action coming from the isomorphism swaps a and $a+b$.

(ii) With the induced $\mathbb{Z}_2$-action $b = a + (a+b)$ and $a(a+b)$ are invariant polynomials. They generate the ring of all invariant polynomials.

(iii) The cohomology $H^i\left((\mathbb{Z}_2)^2, \mathbb{F}_2\right)$ is a $\mathbb{Z}_2$-module and therefore a direct sum of irreducible $\mathbb{Z}_2$-modules. There are only two irreducible $\mathbb{Z}_2$-modules over $\mathbb{F}_2$: the free one $\mathbb{F}_2[\mathbb{Z}_2]$ and the trivial one $\mathbb{F}_2$.

(iv) The isomorphism follows from (iii) and the following two properties of group cohomology [15, Exercise 2.2, page 190] and [6, Corollary 6.6, page 73]. Let M and N be G-modules of a finite group G. Then

(a) $H^*(G, M \oplus N) \cong H^*(G, M) \oplus H^*(G, N)$
(b) M is a free G-module $\Rightarrow$ $H^*(G, M) = H^0(G, M) \cong M^G$.

Applied in our case, this yields

$$\begin{aligned} E_2^{*,i} &=_{\mathrm{ring}} H^*(\mathbb{Z}_2, H^i\left((\mathbb{Z}_2)^2, \mathbb{F}_2\right)) \\ &\cong_{\mathrm{ring}} H^*(\mathbb{Z}_2, \mathbb{F}_2[\mathbb{Z}_2]^{s_{i,1}} \oplus \mathbb{F}_2^{s_{i,2}}) \\ &\cong_{\mathrm{ring}} H^*(\mathbb{Z}_2, \mathbb{F}_2[\mathbb{Z}_2])^{\oplus s_{i,1}} \oplus H^*(\mathbb{Z}_2, \mathbb{F}_2)^{\oplus s_{i,2}} \\ &\cong_{\mathrm{ring}} H^0(\mathbb{Z}_2, \mathbb{F}_2[\mathbb{Z}_2])^{\oplus s_{i,1}} \oplus H^*(\mathbb{Z}_2, \mathbb{F}_2)^{\oplus s_{i,2}} \\ &\cong_{\mathrm{ring}} (\mathbb{F}_2[\mathbb{Z}_2]^{\mathbb{Z}_2})^{\oplus s_{i,1}} \oplus H^*(\mathbb{Z}_2, \mathbb{F}_2)^{\oplus s_{i,2}} \\ &\cong_{\mathrm{ring}} \mathbb{F}_2{}^{\oplus s_{i,1}} \oplus H^*(\mathbb{Z}_2, \mathbb{F}_2)^{\oplus s_{i,2}} \end{aligned}$$

∎

Let the cohomology of the base space of the fibration (20) be denoted by

$$H^*(\mathbb{Z}_2, \mathbb{F}_2) = \mathbb{F}_2[x].$$

The E_2-term (21) can be pictured as in Figure 3.

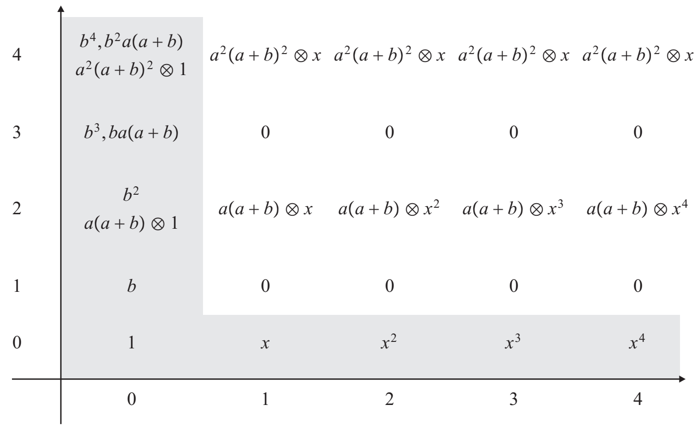

Figure 3: E_2-term

The cohomology of D_8 can be read from the picture. If we denote

$$y := b, \quad w := a(a+b) \tag{22}$$

and keep x as we introduced, then

$$H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, w]/\langle xy \rangle.$$

Also, the restriction homomorphism

$$\mathrm{res}^{D_8}_{H_1} : H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, w]/\langle xy \rangle \quad \rightarrow \quad H^*(H_1, \mathbb{F}_2) = \mathbb{F}_2[a, a+b] \tag{23}$$

can be read off since it is induced by the inclusion of the fibre in the fibration (20). On generators,

$$\mathrm{res}^{D_8}_{H_1}(x) = 0, \ \mathrm{res}^{D_8}_{H_1}(y) = b, \ \mathrm{res}^{D_8}_{H_1}(w) = a(a+b). \tag{24}$$

4.3 The cohomology diagram of subgroups with coefficients in $\mathbb{F}_2$

Let G be a finite group and R an arbitrary ring. Then the diagram $\mathrm{Res}_{(R)} : \mathfrak{G} \to \mathfrak{Ring}$ (covariant functor) defined by

$$\begin{array}{rcl} \mathrm{Ob}(\mathfrak{G}) \ni \ H & \longmapsto & H^*(H, R) \\ (\ H \supseteq K) & \longmapsto & \left(\mathrm{res}^H_K : \ H^*(H, R) \rightarrow \ H^*(K, R)\right) \end{array}$$

is the *cohomology diagram of subgroups* of G with coefficients in the ring R. In this section we assume that $R = \mathbb{F}_2$.

4.3.1 The $\mathbb{Z}_2 \times \mathbb{Z}_2$-diagram

The cohomology of any elementary abelian 2-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a polynomial ring $\mathbb{F}_2\,[x, y]$, $\deg(x) = \deg(y) = 1$. The restrictions to the three subgroups of order 2 are given by all possible projections $\mathbb{F}_2\,[x, y] \to \mathbb{F}_2\,[t]$, $\deg(t) = 1$:

$$(x \mapsto t, y \mapsto 0) \quad \text{or} \quad (x \mapsto 0, y \mapsto t) \quad \text{or} \quad (x \mapsto t, y \mapsto t)\,.$$

Thus the cohomology diagram of the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is

$$
\begin{array}{ccc}
 & \boxed{\begin{array}{c} \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \hline \mathbb{F}_2\,[x, y] \end{array}} & \\
\begin{array}{c} x \mapsto 0 \\ y \mapsto t_1 \\ \swarrow \end{array} & \begin{array}{c} x \mapsto t_2 \\ y \mapsto 0 \\ \downarrow \end{array} & \begin{array}{c} x \mapsto t_3 \\ y \mapsto t_3 \\ \searrow \end{array} \\
\boxed{\begin{array}{c} \mathbb{Z}_2 \\ \hline \mathbb{F}_2\,[t_1\,] \end{array}} & \boxed{\begin{array}{c} \mathbb{Z}_2 \\ \hline \mathbb{F}_2\,[t_2\,] \end{array}} & \boxed{\begin{array}{c} \mathbb{Z}_2 \\ \hline \mathbb{F}_2\,[t_3\,] \end{array}}
\end{array}
\tag{25}
$$

4.3.2 The D_8-diagram

For the dihedral group D_8, from [7] and (23), the two top levels of the diagram can be presented by:

$$
\begin{array}{ccc}
 & \boxed{\begin{array}{c} D_8 \\ \hline \mathbb{F}_2\,[x, y; w]/\langle xy\,\rangle \\ \hline \deg\ : 1, 1, 2 \end{array}} & \\
\begin{array}{c} x \mapsto 0 \\ y \mapsto b \\ w \mapsto a^2\, + ab \\ \swarrow \end{array} & \begin{array}{c} x \mapsto e \\ y \mapsto e \\ w \mapsto u \\ \downarrow \end{array} & \begin{array}{c} x \mapsto d \\ y \mapsto 0 \\ w \mapsto c^2\, + cd \\ \searrow \end{array} \\
\boxed{\begin{array}{c} H_1 \\ \hline \mathbb{F}_2\,[a, b] \\ \hline \deg\ : 1, 1 \end{array}} & \boxed{\begin{array}{c} H_2 \\ \hline \mathbb{F}_2\,[e, u]/\langle e^2\,\rangle \\ \hline \deg\ : 1, 2 \end{array}} & \boxed{\begin{array}{c} H_3 \\ \hline \mathbb{F}_2\,[c, d] \\ \hline \deg\ : 1, 1 \end{array}}
\end{array}
\tag{26}
$$

Let $H^*(K_i, \mathbb{F}_2) = \mathbb{F}_2\,[t_i\,]$, $\deg(t_i) = 1$. From [1, Corollary II.5.7, page 69] the restriction

$$\mathrm{res}^{H_2}_{K_3} : \left(H^*(H_2, \mathbb{F}_2) = \mathbb{F}_2[e, u]\,/\langle e^2\rangle\right) \longrightarrow \left(H^*(K_3, \mathbb{F}_2) = \mathbb{F}_2[t_3]\right)$$

is given by $e \mapsto 0$, $u \mapsto t_3^2$. Thus, the restriction $\mathrm{res}^{D_8}_{K_3}$ is given by $x \mapsto 0$, $y \mapsto 0$, $w \mapsto t_3^2$. Using diagrams (25), (26) with the property (C) from Proposition (3.5 we almost completely reveal the cohomology diagram of subgroups of D_8. The equalities

$$\mathrm{res}^{D_8}_{K_3} = \mathrm{res}^{H_2}_{K_3} \circ \mathrm{res}^{D_8}_{H_2} = \mathrm{res}^{H_1}_{K_3} \circ \mathrm{res}^{D_8}_{H_1} = \mathrm{res}^{H_3}_{K_3} \circ \mathrm{res}^{D_8}_{H_3}$$

imply that

- $\mathrm{res}^{H_1}_{K_3} : (\ H^*(H_1, \mathbb{F}_2) = \mathbb{F}_2\,[a, b]) \longrightarrow (\ H^*(K_3, \mathbb{F}_2) = \mathbb{F}_2\,[t_3\,])$ is given by $a \mapsto t_3,\ b \mapsto 0$,
- $\mathrm{res}^{H_3}_{K_3} : (\ H^*(H_3, \mathbb{F}_2) = \mathbb{F}_2\,[c, d]) \longrightarrow (\ H^*(K_3, \mathbb{F}_2) = \mathbb{F}_2\,[t_3\,])$ is given by $c \mapsto t_3,\ d \mapsto 0$.

$$
\begin{array}{ccccc}
\boxed{\begin{array}{c} H_1 \\ \hline \mathbb{F}_2\,[a, b] \\ \hline \deg\ : 1, 1 \end{array}} & & \boxed{\begin{array}{c} H_2 \\ \hline \mathbb{F}_2\,[e, u]/\langle e^2\,\rangle \\ \hline \deg\ : 1, 2 \end{array}} & & \boxed{\begin{array}{c} H_3 \\ \hline \mathbb{F}_2\,[c, d] \\ \hline \deg\ : 1, 1 \end{array}} \\
 & \begin{array}{c} a \mapsto t_3 \\ b \mapsto 0 \\ \searrow \end{array} & \begin{array}{c} e \mapsto 0 \\ u \mapsto t_3^2 \\ \downarrow \end{array} & \begin{array}{c} c \mapsto t_3 \\ d \mapsto 0 \\ \swarrow \end{array} & \\
 & & \boxed{\begin{array}{c} K_3 \\ \hline \mathbb{F}_2\,[t_3\,] \\ \hline \deg\ : 1 \end{array}} & &
\end{array}
\tag{27}
$$

The cohomology diagram (25) of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the part (27) of the D_8 diagram imply that

- $\mathrm{res}^{H_1}_{K_1} : \mathbb{F}_2\,[a, b] \longrightarrow \mathbb{F}_2\,[t_1\,]$ and $\mathrm{res}^{H_1}_{K_2} : \mathbb{F}_2\,[a, b] \longrightarrow \mathbb{F}_2\,[t_2\,]$ are given by

$$(a \mapsto t_1, b \mapsto t_1 \text{ and } a \mapsto 0, b \mapsto t_2) \ \text{ or } \ (a \mapsto 0, b \mapsto t_1 \text{ and } a \mapsto t_2, b \mapsto t_2)\,,$$

- $\mathrm{res}^{H_3}_{K_4} : \mathbb{F}_2\ [c,d] \longrightarrow \mathbb{F}_2\ [t_4\ ]$ and $\mathrm{res}^{H_3}_{K_5} : \mathbb{F}_2\ [a,b] \longrightarrow \mathbb{F}_2\ [t_5\ ]$ are given by

$$(c \mapsto t_4, d \mapsto t_4 \text{ and } c \mapsto 0, d \mapsto t_5) \text{ or } (c \mapsto 0, d \mapsto t_4 \text{ and } c \mapsto t_5, d \mapsto t_5).$$

Proposition 4.2. *For all* $i \neq 3$, $\mathrm{res}^{D_8}_{K_i}(w) = 0$, *while* $\mathrm{res}^{D_8}_{K_3}(w) \neq 0$.

Proof. The result follows from the diagram (26) discussing both cases (C) and (D). □

Corollary 4.3. *The cohomology of the dihedral group* D_8 *is*

$$H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, w]/\langle xy \rangle$$

where
(a) $x \in H^1(D_8, \mathbb{F}_2)$ *and* $\mathrm{res}^{D_8}_{H_1}(x) = 0$,
(b) $y \in H^1(D_8, \mathbb{F}_2)$ *and* $\mathrm{res}^{D_8}_{H_3}(y) = 0$,
(c) $w \in H^1(D_8, \mathbb{F}_2)$ *and* $\mathrm{res}^{D_8}_{K_1}(w) = \mathrm{res}^{D_8}_{K_2}(w) = \mathrm{res}^{D_8}_{K_4}(w) = \mathrm{res}^{D_8}_{K_5}(w) = 0$ *and* $\mathrm{res}^{D_8}_{K_3}(w) \neq 0$.

Assumption Without losing generality we can assume that

$$\mathrm{res}^{H_1}_{K_1}(a) = t_1, \quad \mathrm{res}^{H_1}_{K_1}(b) = t_1, \quad \mathrm{res}^{H_1}_{K_2}(a) = 0, \quad \mathrm{res}^{H_1}_{K_2}(b) = t_2. \tag{28}$$

4.4 The cohomology ring $H^*(D_8, \mathbb{Z})$

In this section we present the cohomology $H^*(D_8, \mathbb{Z})$ based on:

A. Evans' approach [11, Section 5, pages 191-192], where the concrete generators in $H^*(D_8, \mathbb{Z})$ are identified using the Chern classes of appropriate irreducible complex D_8-representations. We also consider LHS spectral sequences associated with following two extensions

$$1 \to H_1 \to D_8 \to D_8/(\mathbb{Z}_2)^2 \to 1 \qquad \text{and} \qquad 1 \to H_2 \to D_8 \to D_8/\mathbb{Z}_4 \to 1. \tag{29}$$

Unfortunately, the ring structure on E_∞ -terms of these LHS spectral sequences is not going to coincide with the ring structure on $H^*(D_8, \mathbb{Z})$.

B. The Bockstein spectral sequence of the exact couple

$$\begin{tikzcd} H^*(D_8,\mathbb{Z}) \arrow[rr, "p"] & & H^*(D_8,\mathbb{Z}) \arrow[dl, "j"] \\ & H^*(D_8,\mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}$$

where $d_1 = q \circ \delta = \mathrm{Sq}^1 : H^*(D_8, \mathbb{F}_2) \to H^{*+1}(D_8, \mathbb{F}_2)$ is given by $d_1(x) = x^2$, $d_1(y) = y^2$ and $d_1(w) = (x+y)w$ [1, Theorem 2.7. page 127]. This approach allows detection of the ring structure on $H^*(D_8, \mathbb{Z})$.

4.4.1 Evans' view

Let $V^{\mathbb{C}}_{+-} \oplus V^{\mathbb{C}}_{-+} = \mathbb{C} \oplus \mathbb{C}$, $V^{\mathbb{C}}_{--} = \mathbb{C}$ and $U^{\mathbb{C}}_2 = \mathbb{C}$ be complex D_8-representations given by

A. For $(u, v) \in V^{\mathbb{C}}_{+-} \oplus V^{\mathbb{C}}_{-+}$:

$$\varepsilon_1 \cdot (u, v) = (u, -v), \quad \varepsilon_2 \cdot (u, v) = (-u, v), \quad \sigma \cdot (u, v) = (v, u).$$

B. For $u \in V^{\mathbb{C}}_{--}$:

$$\varepsilon_1 \cdot u = -u, \quad \varepsilon_2 \cdot u = -u, \quad \sigma \cdot u = u.$$

C. For $u \in U^{\mathbb{C}}_2$:

$$\varepsilon_1 \cdot u = u, \quad \varepsilon_2 \cdot u = u, \quad \sigma \cdot u = -u.$$

There are isomorphisms of real D_8-representations

$$V_{+-}^{\mathbb{C}} \oplus V_{-+}^{\mathbb{C}} \cong (V_{+-} \oplus V_{-+})^{\oplus 2}, \quad V_{--}^{\mathbb{C}} \cong (V_{--})^{\oplus 2}, \quad U_2^{\mathbb{C}} = (U_2)^{\oplus 2}.$$

Let $\chi_1, \xi \in H^*(D_8, \mathbb{Z})$ be 1-dimensional complex D_8-representations given by character (here we assume identification $c_1 : \mathrm{Hom}(G, U(1)) \to H^2(G, \mathbb{Z})$, [2, page 286]):

$$\begin{array}{lll} \chi_1(\varepsilon_1) = 1, & \chi_1(\varepsilon_2) = 1, & \chi_1(\sigma) = -1, \\ \xi(\varepsilon_1) = -1, & \xi(\varepsilon_2) = -1, & \xi(\sigma) = -1. \end{array}$$

Then $\chi_1 = U_2^{\mathbb{C}}$, $\xi = U_2^{\mathbb{C}} \otimes V_{--}^{\mathbb{C}}$ and consequently

$$\mathrm{c}_1(U_2^{\mathbb{C}}) = \chi_1, \qquad \text{and} \qquad \mathrm{c}_1(U_2^{\mathbb{C}}) + \mathrm{c}_1(V_{--}^{\mathbb{C}}) = \xi. \tag{30}$$

The cohomology $H^*(D_8, \mathbb{Z})$ is given in [11, page 191-192] by

$$H^*(D_8, \mathbb{Z}) = \mathbb{Z}[\xi, \chi_1, \zeta, \chi] \tag{31}$$

where

$$\deg \xi = \deg \chi_1 = 2, \ \deg \zeta = 3, \ \deg \chi = 4$$

and

$$2\xi = 2\chi_1 = 2\zeta = 4\chi = 0, \ \chi_1^2 = \xi \cdot \chi_1, \ \zeta^2 = \xi \cdot \chi. \tag{32}$$

There are three 1-dimensional irreducible complex representations of D_8:

$$1, \ \xi = U_2^{\mathbb{C}} \otimes V_{--}^{\mathbb{C}}, \ \chi_1 = U_2^{\mathbb{C}}, \ \xi \otimes \chi_1 = V_{--}^{\mathbb{C}},$$

and one 2-dimensional complex representation which is denoted by ρ in [11, page 191-192]:

$$\rho = V_{+-}^{\mathbb{C}} \oplus V_{-+}^{\mathbb{C}}.$$

It is computed in [11, page 191-192] that

$$\mathrm{c}(V_{+-}^{\mathbb{C}} \oplus V_{-+}^{\mathbb{C}}) = 1 + \xi + \chi \qquad \text{and} \qquad \mathrm{c}_2(V_{+-}^{\mathbb{C}} \oplus V_{-+}^{\mathbb{C}}) = \chi. \tag{33}$$

The relations (30) and (33) along with Proposition 3.11 imply the following statement.

Proposition 4.4. $\mathrm{Index}_{D_8,\mathbb{Z}} S(V_{--}^{\mathbb{C}}) = \langle \xi + \chi_1 \rangle$, $\mathrm{Index}_{D_8,\mathbb{Z}} S(U_2^{\mathbb{C}}) = \langle \chi_1 \rangle$, $\mathrm{Index}_{D_8,\mathbb{Z}} S(V_{+-}^{\mathbb{C}} \oplus V_{-+}^{\mathbb{C}}) = \langle \chi \rangle$.

Before proceeding to the Bockstein spectral sequence approach we give descriptions of the E_2-terms of two LHS spectral sequences. Even it is not an easy consequence, it can be proved that both spectral sequences stabilize and that $E_2 = E_\infty$.

LHS spectral sequences of the extension $1 \to H_1 \to D_8 \to D_8/H_1 \to 1$**.** The LHS spectral sequence of this extension (21) allowed computation of the cohomology ring $H^*(D_8, \mathbb{F}_2)$ with $\mathbb{F}_2$ coefficients. If we now consider $\mathbb{Z}$ coefficients, then the E_2-term has form

$$E_2^{p,q} = H^p(D_8/H_1, H^q(H_1, \mathbb{Z})) \cong H^p(\mathbb{Z}_2, H^q((\mathbb{Z}_2)^2, \mathbb{Z})). \tag{34}$$

The spectral sequence converges to the graded group $\mathrm{Gr}(H^{p+q}(D_8, \mathbb{Z}))$ associated with $H^{p+q}(D_8, \mathbb{Z})$ appropriately filtered. To present the E_2-term we chose generators of $H^*(H_1, \mathbb{Z})$ consistently with choices made in Lemma 4.1. Let $j : \mathbb{Z} \to \mathbb{F}_2$ be a ring morphism and $j^* : H^*(D_8, \mathbb{Z}) \to H^*(D_8, \mathbb{F}_2)$ the induced map in cohomology. Consider the following presentation of the H_1 cohomology ring:

$$H^*(H_1, \mathbb{Z}) = \mathbb{Z}[\alpha, \alpha + \beta] \otimes \mathbb{Z}[\mu] \tag{35}$$

where

A. $\deg(\alpha) = \deg(\beta) = 2$, $\deg(\mu) = 3$;

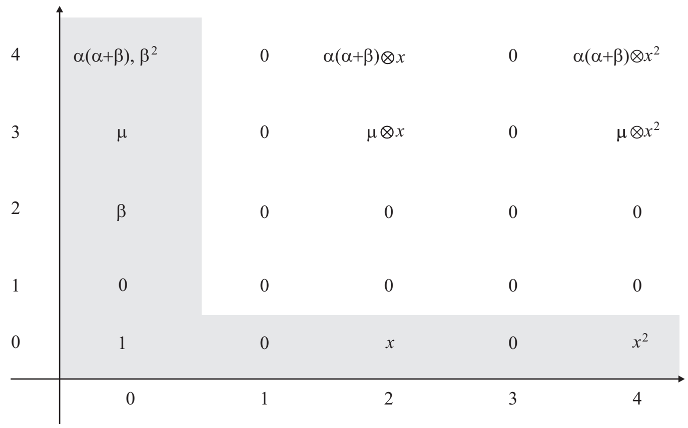

Figure 4: E_2-term of extension $1 \to H_1 \to D_8 \to D_8/H_1 \to 1$.

B. $2\alpha = 2\beta = 2\mu = 0$ and $\mu^2 = \alpha\beta(\alpha+\beta)$;

C. σ action on $H^*(H_1, \mathbb{Z})$ is given by $\sigma \cdot \alpha = \alpha + \beta$ and $\sigma \cdot \mu = \mu$;

D. $j^*(\alpha) = a^2$, $j^*(\beta) = b^2$, $j^*(\mu) = ab(a+b)$.

Now the E_2-term (Figure 4) is given by

$$E_2^{p,q} \cong H^p(\mathbb{Z}_2, H^q((\mathbb{Z}_2)^2, \mathbb{Z})) \cong \begin{cases} H^p(\mathbb{Z}_2, \mathbb{Z}), & q = 0 \\ 0, & q = 1 \\ H^p(\mathbb{Z}_2, \mathbb{F}_2[\mathbb{Z}_2]), & q = 2 \\ H^p(\mathbb{Z}_2, \mathbb{F}_2), & q = 3 \\ ..., & q > 3 \end{cases}.$$

The morphism of LHS spectral sequences of the extension $1 \to H_1 \to D_8 \to D_8/H_1 \to 1$ induced by the ring map $j : \mathbb{Z} \to \mathbb{F}_2$ (Proposition 3.5 E.3) gives a proof that $E_2 = E_\infty$ for $\mathbb{Z}$ coefficients. The ring structures on E_∞ and $H^*(D_8, \mathbb{Z})$ do not coincide, moreover there is no element in E_∞ of exponent 4. One thing is clear, the element μ in $E_2 = E_\infty$-term coincides with the element ζ in the Evans' presentation (31) of $H^*(D_8, \mathbb{Z})$.

LHS spectral sequences of the extension $1 \to H_2 \to D_8 \to D_8/H_2 \to 1$**.** The E_2-term has the form:

$$E_2^{p,q} = H^p(D_8/H_2, H^q(H_2, \mathbb{Z})) \cong H^p(\mathbb{Z}_2, H^q(\mathbb{Z}_4, \mathbb{Z})) \cong \begin{cases} H^p(\mathbb{Z}_2, \mathbb{Z}), & q = 0 \\ 0, & q \text{ odd} \\ H^p(\mathbb{Z}_2, \mathcal{Z}_4), & q \text{ even and } 4 \nmid q \\ H^p(\mathbb{Z}_2, \mathbb{Z}_4), & q > 0 \text{ even and } 4 | q \end{cases},$$

where $\mathcal{Z}_4 \cong \mathbb{Z}_4$ is a non-trivial $\mathbb{Z}_2$-module. Using [6, Example 2, page 58-59] the E_2-term gets the shape as in the Figure 5. This picture gives just two hints: there might be elements of exponent 4 in the cohomology $H^*(D_8, \mathbb{Z})$ and definitely there is only one element ζ of degree 3 from the Evans' presentation.

Conclusion. The LHS spectral sequences of different extensions gives an incomplete picture of the cohomology ring with integer coefficients, $H^*(D_8, \mathbb{Z})$. Therefore, for the purposes of the computations with $\mathbb{Z}$ coefficients we use Bockstein spectral sequence utilizing results obtained by LHS spectral sequence with $\mathbb{F}_2$ coefficients. Presentations of these two spectral sequences are going to be used in the description of the restriction diagram in Section 4.5.

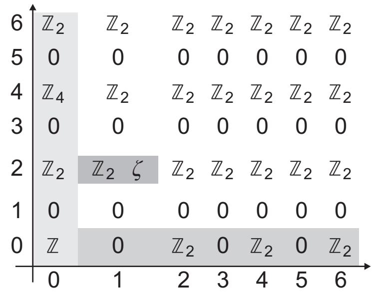

Figure 5: E_2-term of extension $1 \to H_2 \to D_8 \to D_8/H_2 \to 1$.

4.4.2 The Bockstein spectral sequence view

Let G be a finite group. The exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{F}_2 \to 0$ induces a long exact sequence in group cohomology, or an exact couple

$$\begin{tikzcd}[column sep=small] H^*(G,\mathbb{Z}) \arrow[rr, "\times 2"] & & H^*(G,\mathbb{Z}) \arrow[dl, "j"] \\ & H^*(G,\mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}.$$

The spectral sequence of this exact couple is the Bockstein spectral sequence. It converges to

$$(H^*(G,\mathbb{Z})/\text{torsion}) \otimes \mathbb{F}_2$$

which in the case of the finite group G is just $\mathbb{F}_2$ in dimension 0. Here "torsion" means $\mathbb{Z}$-torsion. The first differential $d_1 = j \circ \delta$ is the Bockstein homomorphism and in this case coincides with the first Steenrod square $\mathrm{Sq}^1 : H^*(G,\mathbb{F}_2) \to H^{*+1}(G,\mathbb{F}_2)$.
Let H be a subgroup of G. The restriction res^G_H commutes with the maps in the exact couples associated to groups G and H and therefore induces a morphism of Bockstein spectral sequences [8, page 109 before 5.7.6].

Consider two Bockstein spectral sequences associated with D_8 and its subgroup $H_2 \cong \mathbb{Z}_4$.

A. **Group** D_8. The exact couple is

$$\begin{tikzcd}[column sep=small] H^*(D_8,\mathbb{Z}) \arrow[rr, "\times 2"] & & H^*(D_8,\mathbb{Z}) \arrow[dl, "j"] \\ & H^*(D_8,\mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd} \tag{36}$$

and $d_1 = j \circ \delta = \mathrm{Sq}^1$ is given by $d_1(x) = x^2$, $d_1(y) = y^2$ and $d_1(w) = (x+y)w$, [1, Theorem 2.7. page 127]. The derived couple is

$$\begin{tikzcd}[column sep=small] 2 \cdot H^*(D_8,\mathbb{Z}) \arrow[rr, "\times 2"] & & 2 \cdot H^*(D_8,\mathbb{Z}) \arrow[dl, "j_1"] \\ & \langle x^2, y^2, xw, yw, w^2\rangle/\langle x^2, y^2, xw+yw\rangle \arrow[ul, "\delta_1"] & \end{tikzcd}.$$

Then by [8, Remark 5.7.4, page 108] there are elements $\mathcal{X}, \mathcal{Y} \in H^2(D_8,\mathbb{Z})$, $\mathcal{M} \in H^3(D_8,\mathbb{Z})$ of exponent 2 such that $j(\mathcal{X}) = x^2$, $j(\mathcal{Y}) = y^2$, $j(\mathcal{M}) = (x+y)w$ and $\mathcal{X}\mathcal{Y} = 0$.

B. **Group** $\mathbb{Z}_4$. The exact couple is

$$\begin{tikzcd}[column sep=small] H^*(\mathbb{Z}_4,\mathbb{Z}) \arrow[rr, "\times 2"] & & H^*(\mathbb{Z}_4,\mathbb{Z}) \arrow[dl, "j"] \\ & H^*(\mathbb{Z}_4,\mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}. \tag{37}$$

Since $H^*(\mathbb{Z}_4,\mathbb{Z})=\mathbb{Z}[U]/\langle 4U\rangle$, $\deg U=2$ and $H^*(\mathbb{Z}_4,\mathbb{F}_2)=\mathbb{F}_2\ [e,u]/\langle e^2\rangle$, $\deg e=1$, $\deg u=2$, the unrolling of the exact couple to a long exact sequence [6, Proposition 6.1, page 71]

$$\begin{array}{cccccccc}
0 & \rightarrow & \begin{matrix} H^0(\mathbb{Z}_4,\mathbb{Z}) \\ \mathbb{Z}\ \ ,\ 1 \end{matrix} & \xrightarrow{\times 2} & \begin{matrix} H^0(\mathbb{Z}_4,\mathbb{Z}) \\ \mathbb{Z}\ \ ,\ 1 \end{matrix} & \rightarrow & \begin{matrix} H^0(\mathbb{Z}_4,\mathbb{F}_2) \\ \mathbb{F}_2\ \ ,\ 1 \end{matrix} \\
 & \xrightarrow{\delta} & \begin{matrix} H^1(\mathbb{Z}_4,\mathbb{Z}) \\ 0 \end{matrix} & \xrightarrow{\times 2} & \begin{matrix} H^1(\mathbb{Z}_4,\mathbb{Z}) \\ 0 \end{matrix} & \rightarrow & \begin{matrix} H^1(\mathbb{Z}_4,\mathbb{F}_2) \\ \mathbb{F}_2\ \ ,\ e \end{matrix} \\
 & \xrightarrow{\delta} & \begin{matrix} H^2(\mathbb{Z}_4,\mathbb{Z}) \\ \mathbb{Z}_4\ \ ,\ U \end{matrix} & \xrightarrow{\times 2} & \begin{matrix} H^2(\mathbb{Z}_4,\mathbb{Z}) \\ \mathbb{Z}_4\ \ ,\ U \end{matrix} & \rightarrow & \begin{matrix} H^2(\mathbb{Z}_4,\mathbb{F}_2) \\ \mathbb{F}_2\ \ ,\ u \end{matrix} \\
 & \xrightarrow{\delta} & \begin{matrix} H^3(\mathbb{Z}_4,\mathbb{Z}) \\ 0 \end{matrix} & \xrightarrow{\times 2} & \begin{matrix} H^3(\mathbb{Z}_4,\mathbb{Z}) \\ 0 \end{matrix} & \rightarrow & \begin{matrix} H^3(\mathbb{Z}_4,\mathbb{F}_2) \\ \mathbb{F}_2\ \ ,\ eu \end{matrix} \\
 & \xrightarrow{\delta} & \begin{matrix} H^4(\mathbb{Z}_4,\mathbb{Z}) \\ U^2 \end{matrix} & & \dots & &
\end{array}$$

allow us to show that for $j\geq 0$:

$$\delta(u^i)=0 \text{ and } \delta(eu^i)=2U^{i+1}.$$

Thus $d_1=0$ and the derived couple is

$$\begin{tikzcd}
2\cdot H^*(\mathbb{Z}_4,\mathbb{Z}) \arrow[rr, "\times 2"] & & 2\cdot H^*(\mathbb{Z}_4,\mathbb{Z}) \arrow[dl, "j_1"] \\
 & H^*(\mathbb{Z}_4,\mathbb{F}_2) \arrow[ul, "\delta_1"] &
\end{tikzcd}.$$

Moreover, by definition of the differential of derived couple we have that

$$d_2(u^i)=0 \text{ and } d_2(eu^i)=u^{i+1}.$$

The restriction map $\operatorname{res}^{D_8}_{H_2}: H^*(D_8,\mathbb{F}_2)\rightarrow H^*(H_2,\mathbb{F}_2)$ is determined in the restriction diagram (26). Therefore, the morphism between spectral sequences induced by the restriction $\operatorname{res}^{D_8}_{H_2}$ implies that:

$$\operatorname{res}^{D_8}_{H_2}(d_2[xw])=d_2\left(\operatorname{res}^{D_8}_{H_2}[xw]\right)=d_2(eu)=u^2$$

and consequently

$$d_2[xw]=[w^2]$$

Here [.] denotes the class in the quotient $\langle x^2,y^2,xw,yw,w^2\rangle/\langle x^2,y^2,xw+yw\rangle$. Thus, by [8, Remark 5.7.4, page 108] there is an element $\mathcal{W}\in H^4(D_8,\mathbb{Z})$ of exponent 4 such that $j(\mathcal{W})=w^2$ and $\mathcal{M}^2=\mathcal{W}(\mathcal{X}+\mathcal{Y})$. The second derived couple of (36) stabilizes. Thus the cohomology ring $H^*(D_8,\mathbb{Z})$ and the map $j: H^*(D_8,\mathbb{Z})\longrightarrow H^*(D_8,\mathbb{F}_2)$ are described.

Theorem 4.5. *The cohomology ring $H^*(D_8,\mathbb{Z})$ can be presented by*

$$H^*(D_8,\mathbb{Z})=\mathbb{Z}[\mathcal{X},\mathcal{Y},\mathcal{M},\mathcal{W}]$$

where

$$\deg\mathcal{X}=\deg\mathcal{Y}=2,\ \ \deg\mathcal{M}=3,\ \ \deg\mathcal{W}=4$$

and

$$2\mathcal{X}=2\mathcal{Y}=2\mathcal{M}=4\mathcal{W}=0,\ \mathcal{X}\mathcal{Y}=0,\ \mathcal{M}^2=\mathcal{W}(\mathcal{X}+\mathcal{Y}). \tag{38}$$

The map $j: H^*(D_8,\mathbb{Z})\longrightarrow H^*(D_8,\mathbb{F}_2)$, induced by the reduction of coefficients $\mathbb{Z}\rightarrow\mathbb{F}_2$, is given by

$$\mathcal{X}\mapsto x^2\ ,\qquad \mathcal{Y}\mapsto y^2\ ,\qquad \mathcal{M}\mapsto w(x+y)\ ,\qquad \mathcal{W}\mapsto w^2. \tag{39}$$

Remark 4.6. The correspondence between Evans' and Bockstein view is given by

$$\mathcal{X}\leftrightarrow\chi_1\ ,\qquad \mathcal{Y}\leftrightarrow\xi+\chi_1\ ,\qquad \mathcal{M}\leftrightarrow\zeta\ ,\qquad \mathcal{W}\leftrightarrow\chi \tag{40}$$

4.5 The D_8-diagram with coefficients in $\mathbb{Z}$

Let G be a finite group and R and S rings. A ring homomorphism $\phi : R \to S$ induces a morphism of diagrams (natural transformation of covariant functors) $\Phi :\mathrm{Res}_{(R)} \to \mathrm{Res}_{(S)}$. Particularly, in this section, as a tool for the reconstruction of the diagram $\mathrm{Res}_{(\mathbb{Z})}$, we use the diagram morphism $J :\mathrm{Res}_{(\mathbb{Z})} \to \mathrm{Res}_{(\mathbb{F}_2)}$ induced by the coefficient reduction homomorphism $j : \mathbb{Z} \to \mathbb{F}_2$.

4.5.1 The $\mathbb{Z}_2 \times \mathbb{Z}_2$-diagram

The cohomology restriction diagram $\mathrm{Res}_{(\mathbb{F}_2)}$ of elementary abelian 2-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the diagram (25). Using the presentation of cohomology $H^*(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z})$ and the homomorphism $H^*(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}) \to H^*(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{F}_2)$ given in Example 3.10 we can reconstruct the restriction diagram $\mathrm{Res}_{(\mathbb{Z})}$:

$$
\begin{array}{ccc}
 & \begin{array}{|l|l|}\hline \mathbb{Z}_2 \times \mathbb{Z}_2 & \mathbb{Z}[\tau_1, \tau_2] \otimes \mathbb{Z}[\mu] \\ \hline \multicolumn{2}{|l|}{\deg \tau_1 = \deg \tau_2 = 2,\ \deg \mu = 3} \\ \hline \multicolumn{2}{|l|}{2\tau_1 = 2\tau_2 = 2\mu = 0,} \\ \hline \multicolumn{2}{|l|}{\mu^2 = \tau_1\tau_2(\tau_1 + \tau_2)} \\ \hline \end{array} & \\
\begin{array}{c}\tau_1 \mapsto 0, \tau_2 \mapsto \theta_1 \\ \mu \mapsto 0 \quad \swarrow\end{array} & \begin{array}{c}\tau_1 \mapsto \theta_2 \qquad \mu \mapsto 0 \\ \tau_2 \mapsto 0 \quad \downarrow\end{array} & \begin{array}{c}\tau_1 \mapsto \theta_3, \tau_2 \mapsto \theta_3 \\ \searrow \quad \mu \mapsto 0\end{array} \\
\begin{array}{|l|l|}\hline \mathbb{Z}_2 & \mathbb{Z}[\theta_1] \\ \hline \multicolumn{2}{|l|}{\deg \theta_1 = 2} \\ \hline \multicolumn{2}{|l|}{2\theta_1 = 0} \\ \hline \end{array} & \begin{array}{|l|l|}\hline \mathbb{Z}_2 & \mathbb{Z}[\theta_2] \\ \hline \multicolumn{2}{|l|}{\deg \theta_2 = 2} \\ \hline \multicolumn{2}{|l|}{2\theta_2 = 0} \\ \hline \end{array} & \begin{array}{|l|l|}\hline \mathbb{Z}_2 & \mathbb{Z}[\theta_3] \\ \hline \multicolumn{2}{|l|}{\deg \theta_3 = 2} \\ \hline \multicolumn{2}{|l|}{2\theta_3 = 0} \\ \hline \end{array}
\end{array}
\tag{41}
$$

4.5.2 The D_8-diagram

In the similar fashion, using:

- the D_8 restriction diagram (26) and (27) with $\mathbb{F}_2$ coefficients,
- the $\mathbb{Z}_2 \times \mathbb{Z}_2$ restriction diagram (41) with $\mathbb{Z}$ coefficients,
- the presentation of cohomology $H^*(H_1, \mathbb{Z})$ given in (35),
- the Bockstein presentation (**??**) of $H^*(D_8, \mathbb{Z})$,
- glimps of the restriction maps $\mathrm{res}^{D_8}_{H_1}$ and $\mathrm{res}^{D_8}_{H_2}$ obtained from the $E_2 = E_\infty$ terms of the LHS spectral sequences Figure 4 and Figure 5, and
- the homomorphism $j : H^*(D_8, \mathbb{Z}) \to H^*(D_8, \mathbb{F}_2)$ described in (39),

we can reconstruct the restriction diagram of D_8 with $\mathbb{Z}$ coefficients.

$$
\begin{array}{ccc}
 & \begin{array}{|l|l|}\hline D_8 & \mathbb{Z}[\mathcal{X}, \mathcal{Y}, \mathcal{M}, \mathcal{W}] \\ \hline \multicolumn{2}{|l|}{\deg : 2,\ 2,\ 3,\ 4} \\ \hline \multicolumn{2}{|l|}{2\mathcal{X} = 2\mathcal{Y} = 2\mathcal{M} = 4\mathcal{W} = 0,} \\ \hline \multicolumn{2}{|l|}{\mathcal{X}\mathcal{Y} = 0,\ \mathcal{M}^2 = \mathcal{W}(\mathcal{X} + \mathcal{Y})} \\ \hline \end{array} & \\
\begin{array}{c}\mathcal{X} \mapsto 0,\ \mathcal{Y} \mapsto \beta, \mathcal{M} \mapsto \mu \\ \mathcal{W} \mapsto \alpha(\alpha + \beta) \quad \swarrow\end{array} & \begin{array}{c}\mathcal{X} \mapsto 2U \qquad \mathcal{Y} \mapsto 2U \\ \mathcal{M} \mapsto 0 \quad \downarrow \quad \mathcal{W} \mapsto U^2\end{array} & \begin{array}{c}\mathcal{X} \mapsto \delta,\ \mathcal{Y} \mapsto 0, \mathcal{M} \mapsto \eta \\ \searrow \quad \mathcal{W} \mapsto \gamma(\gamma + \delta)\end{array} \\
\begin{array}{|l|l|}\hline H_1 & \mathbb{Z}[\alpha, \alpha + \beta, \mu] \\ \hline \multicolumn{2}{|l|}{\deg : 2,\ 2,\ 3,} \\ \hline \multicolumn{2}{|l|}{2\alpha = 2\beta = 2\mu = 0,} \\ \hline \multicolumn{2}{|l|}{\mu^2 = \alpha\beta(\alpha + \beta)} \\ \hline \end{array} & \begin{array}{|l|l|}\hline H_2 & \mathbb{Z}[U] \\ \hline \multicolumn{2}{|l|}{\deg : 2} \\ \hline \multicolumn{2}{|l|}{4U = 0,} \\ \hline \end{array} & \begin{array}{|l|l|}\hline H_3 & \mathbb{Z}[\gamma, \gamma + \delta, \eta] \\ \hline \multicolumn{2}{|l|}{\deg : 2,\ 2,\ 3,} \\ \hline \multicolumn{2}{|l|}{2\gamma = 2\delta = 2\eta = 0,} \\ \hline \multicolumn{2}{|l|}{\eta^2 = \gamma\delta(\gamma + \delta)} \\ \hline \end{array} \\
\begin{array}{c}\alpha \mapsto \theta_3,\ \beta \mapsto 0 \\ \mu \mapsto 0 \quad \searrow\end{array} & \begin{array}{c}U \mapsto \theta_3 \\ \downarrow\end{array} & \begin{array}{c}\gamma \mapsto \theta_3, \delta \mapsto 0 \\ \swarrow \quad \eta \mapsto 0\end{array} \\
 & \begin{array}{|l|l|l|}\hline K_3 & \mathbb{Z}[\theta_3] & \deg \theta_3 = 2 \\ \hline \end{array} &
\end{array}
\tag{42}
$$

Now the reconstruction of the diagram morphism $J :\mathrm{Res}_{(\mathbb{Z})} \to \mathrm{Res}_{(\mathbb{F}_2)}$ induced by the coefficient reduction homomorphism $j : \mathbb{Z} \to \mathbb{F}_2$ is just the routine exercise.

5 $\mathbf{Index}_{D_8,\mathbb{F}_2}\mathbf{S}(\mathbf{R}_4^{\oplus j})$

In this section we show the following equality

$$\mathrm{Index}_{D_8,\mathbb{F}_2}S(R_4^{\oplus j}) = \mathrm{Index}^{3j+1}_{D_8,\mathbb{F}_2}S(R_4^{\oplus j}) = \langle w^j y^j\rangle.$$

The D_8-representation $R_4^{\oplus j}$ can be decomposed into a sum of irreducibles in the following way

$$R_4 = (V_{-+}\oplus V_{+-})\oplus V_{--} \;\Rightarrow\; R_4^{\oplus j} = (V_{-+}\oplus V_{+-})^{\oplus j}\oplus V_{--}^{\oplus j}$$

where $V_{-+}\oplus V_{+-}$ is a 2-dimensional irreducible D_8-representation. Since in this section the $\mathbb{F}_2$ coefficients are assumed, Proposition 3.12 implies that computing the indexes of the spheres $S(V_{-+}\oplus V_{+-})$ and $S(V_{--})$ suffices. The strategy employed uses Proposition 3.7 and the following particular facts.
A. Let $X = S(T)$ for some D_8-representation T. Then the E_2-term of the Serre spectral sequence associated to $\mathrm{E}D_8\times_{D_8}X$ is

$$E_2^{p,q} = H^p(D_8,\mathbb{F}_2)\otimes H^q(X,\mathbb{F}_2). \tag{43}$$

The local coefficients are trivial since X is a sphere and the coefficients are $\mathbb{F}_2$. Since only $\partial_{\dim T,\mathbb{F}_2}$ may be $\neq 0$, from the multiplicativity property of the spectral sequence it follows that

$$\mathrm{Index}_{D_8,\mathbb{F}_2}X = \langle\partial^{0,\dim V-1}_{\dim V,\mathbb{F}_2}(1\otimes l)\rangle$$

where $l\in H^{\dim V-1}(X,\mathbb{F}_2)$ is the generator. Therefore, $\mathrm{Index}_{D_8,\mathbb{F}_2}(X) = \mathrm{Index}^{\dim V+1}_{D_8,\mathbb{F}_2}(X)$.
B. For any subgroup H of D_8, with some abuse of notation,

$$\Gamma^{\dim V,0}_{\dim V}\circ\partial^{0,\dim V-1}_{\dim V,\mathbb{F}_2}(1\otimes l) = \partial^{0,\dim V-1}_{\dim V,\mathbb{F}_2}\circ\Gamma^{0,\dim V-1}_{\dim V}(1\otimes l), \tag{44}$$

where Γ denotes the restriction morphism of Serre spectral sequences introduced in Proposition 3.5(D). Therefore, for every subgroup H of D_8 we get

$$\mathrm{Index}_{D_8,\mathbb{F}_2}X = \langle a\rangle,\quad \mathrm{Index}_{H,\mathbb{F}_2}X = \langle a_H\rangle \implies \mathrm{res}^G_K(a) = a_H.$$

Particularly, if $a_H\neq 0$ then $a\neq 0$.

Our computation of $\mathrm{Index}_{D_8,\mathbb{F}_2}X$ for $X = S(V_{-+}\oplus V_{+-})$ and $X = \mathrm{Index}_{D_8,\mathbb{F}_2}S(V_{--})$ has two steps:

- compute $\mathrm{Index}_{H,\mathbb{F}_2}X = \langle a_H\rangle$ for all proper subgroups H of D_8,
- search for an element $a\in H^*(D_8,\mathbb{F}_2)$ such that for every computed a_H

$$\mathrm{res}^G_K(a) = a_H.$$

5.1 $\mathrm{Index}_{D_8,\mathbb{F}_2}S(V_{-+}\oplus V_{+-}) = \langle w\rangle$

Proposition 3.13 and the properties of D_8 acting on $V_{-+}\oplus V_{+-}$ provide the following information

$$\mathrm{Index}_{H_1,\mathbb{F}_2}S(V_{-+}\oplus V_{+-}) = \begin{cases}\langle a(a+b)\rangle & \text{or}\\ \langle b(a+b)\rangle & \text{or}\\ \langle ab\rangle.\end{cases}$$

Since initially we do not know which of possible generators a, b, $a+b$ of $\mathbb{F}_2[a,b]$ correspond to the generators ε_1, ε_2, $\varepsilon_1\varepsilon_2$, we have to take all three possibilities into account. Similarly

$$\mathrm{Index}_{H_3,\mathbb{F}_2}S(V_{-+}\oplus V_{+-}) = \begin{cases}\langle c(c+d)\rangle & \text{or}\\ \langle d(c+d)\rangle & \text{or}\\ \langle cd\rangle.\end{cases}$$

Furthermore,

$$\begin{array}{lcl}
\varepsilon_1 \text{ acts trivially on } V_{+-} & \Rightarrow & \mathrm{Index}_{K_1,\mathbb{F}_2}S(V_{-+}\oplus V_{+-}) = 0\\
\varepsilon_2 \text{ acts trivially on } V_{-+} & \Rightarrow & \mathrm{Index}_{K_2,\mathbb{F}_2}S(V_{-+}\oplus V_{+-}) = 0\\
\sigma \text{ acts trivially on } \{(x,x)\in V_{-+}\oplus V_{+-}\} & \Rightarrow & \mathrm{Index}_{K_4,\mathbb{F}_2}S(V_{-+}\oplus V_{+-}) = 0\\
\varepsilon_1\varepsilon_2\sigma \text{ acts trivially on } \{(x,-x)\in V_{-+}\oplus V_{+-}\} & \Rightarrow & \mathrm{Index}_{K_5,\mathbb{F}_2}S(V_{-+}\oplus V_{+-}) = 0
\end{array}$$

The only element inside $H^2(D_8, \mathbb{F}_2)$ satisfying all requirements of commutativity with restrictions is w. Hence,

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S(V_{-+} \oplus V_{+-}) = \langle w \rangle \tag{45}$$

Remark 5.1. The side information coming from this computation is that generators ε_1 and ε_2 of the group H_1 correspond to generators a and $a+b$ in the cohomology ring $H^*(H_1, F_2)$. This correspondence suggested the choice of generators in Lemma 4.1(i).

5.2 $\mathrm{Index}_{D_8,\mathbb{F}_2} S(V_{--}) = \langle y \rangle$

Again, V_{--} is a concrete D_8-representation, and from Proposition 3.13:

$$\mathrm{Index}_{H_1,\mathbb{F}_2} S(V_{--}) = \begin{cases} \langle a+b \rangle, & \text{or} \\ \langle a+(a+b) \rangle, & \text{or} \\ \langle b+(a+b) \rangle. & \end{cases}$$

Again, we allow all three possibilities since we do not know the correspondence between generators of H_1 and the chosen generators of $H^*(H_q, \mathbb{F}_2)$. Furthermore, since K_1 and K_2 act nontrivially on V_{--},

$$\mathrm{Index}_{K_1,\mathbb{F}_2} S(V_{--}) = \langle t_1 \rangle, \quad \mathrm{Index}_{K_2,\mathbb{F}_2} S(V_{--}) = \langle t_2 \rangle \ .$$

On the other hand, H_3 acts trivially on $S(V_{--})$ and so

$$\mathrm{Index}_{H_3,\mathbb{F}_2} S(V_{--}) = 0.$$

By commutativity of the restriction diagram, or since the groups K_3, K_4 and K_5 act trivially on $V_{(1,1)}$, it follows that

$$\mathrm{Index}_{K_3,\mathbb{F}_2} S(V_{--}) \ = \ \mathrm{Index}_{K_4,\mathbb{F}_2} S(V_{--}) \ = \ \mathrm{Index}_{K_5,\mathbb{F}_2} S(V_{--}) \ = \ 0.$$

The only element satisfying the commutativity requirements is $y \in H^1(D_8, \mathbb{F}_2)$, so

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S(V_{--}) = \langle y \rangle. \tag{46}$$

Remark 5.2. From the previous remark the fact $\mathrm{Index}_{H_1,\mathbb{F}_2} S(V_{--}) = \langle b \rangle = \langle a+(a+b) \rangle$ follows directly. Alternatively, equation (46) is the consequence of (11) and (26).

5.3 $\mathrm{Index}_{D_8,\mathbb{F}_2} S(R_4^{\oplus j}) = \langle y^j w^j \rangle$

From Proposition 3.12 we get that

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S(R_4^{\oplus j}) = \mathrm{Index}_{D_8,\mathbb{F}_2} S((V_{-+} \oplus V_{+-})^{\oplus j} \oplus V_{--}^{\oplus j}) = \langle y^j w^j \rangle.$$

Remark 5.3. In the same way we can compute that

$$\mathrm{Index}_{D_8,\mathbb{F}_2} (U_2) = \langle x \rangle. \tag{47}$$

Therefore $\mathrm{Index}_{D_8,\mathbb{F}_2} (U_2 \oplus R_4^{\oplus j}) = 0$. This means that on the join CS/TM scheme the Fadell–Husseini index theory with $\mathbb{F}_2$ coefficients does not give any non-trivial result, since 0 is an element of any ideal.

6 $\mathbf{Index}_{D_8,\mathbb{Z}} \mathbf{S}(\mathbf{R}_4^{\oplus j})$

In this section we show that

$$\mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) = \mathrm{Index}^{3j+1}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) = \begin{cases} \langle \mathcal{Y}^{\frac{j}{2}} \mathcal{W}^{\frac{j}{2}} \rangle, & \text{for } j \text{ even} \\ \langle \mathcal{Y}^{\frac{j+1}{2}} \mathcal{W}^{\frac{j-1}{2}} \mathcal{M}, \mathcal{Y}^{\frac{j+1}{2}} \mathcal{W}^{\frac{j+1}{2}} \rangle, & \text{for } j \text{ odd} \end{cases} . \tag{48}$$

6.1 The case when j is even

We give two proofs of the equation (48) in the case when j is even.

Method 1: According to definition of the complex D_8-representations $V_{+-}^{\mathbb{C}} \oplus V_{-+}^{\mathbb{C}}$ and $V_{--}^{\mathbb{C}}$, in Section 4.4.1, we have an isomorphism of real D_8-representations

$$R_4^{\oplus j} = (V_{-+} \oplus V_{+-})^{\oplus j} \oplus V_{--}^{\oplus j} \cong \left(V_{+-}^{\mathbb{C}} \oplus V_{-+}^{\mathbb{C}}\right)^{\oplus \frac{j}{2}} \oplus \left(V_{--}^{\mathbb{C}}\right)^{\oplus \frac{j}{2}}.$$

Thus by Propositions 3.11 and 3.12, properties of Chern classes [2, (5) page 286], equations (30) and (33) we have that

$$\begin{aligned} \mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) &= \langle \mathrm{c}_{\frac{3j}{2}}\left(\left(V_{+-}^{\mathbb{C}} \oplus V_{-+}^{\mathbb{C}}\right)^{\oplus \frac{j}{2}} \oplus \left(V_{--}^{\mathbb{C}}\right)^{\oplus \frac{j}{2}}\right)\rangle = \langle \mathrm{c}_2\left(V_{+-}^{\mathbb{C}} \oplus V_{-+}^{\mathbb{C}}\right)^{\frac{j}{2}} \cdot \mathrm{c}_1\left(V_{--}^{\mathbb{C}}\right)^{\frac{j}{2}}\rangle \\ &= \langle \chi^{\frac{j}{2}}\left(\xi + \chi_1\right)^{\frac{j}{2}}\rangle. \end{aligned}$$

The correspondence between Evans' and Bockstein view implies the statement.

Method 2: The group D_8 acts trivially on the cohomology $H^*(S(R_4^{\oplus j}), \mathbb{Z})$. Then the E_2-term of the Serre spectral sequence associated to $\mathrm{E}D_8 \times_{D_8} S(R_4^{\oplus j})$ is a tensor product

$$E_2^{p,q} = H^p(D_8, \mathbb{Z}) \otimes H^q(S(R_4^{\oplus j}), \mathbb{Z}).$$

Since only $\partial_{3j,\mathbb{Z}}$ may be $\neq 0$, the multiplicativity property of the spectral sequence implies that

$$\mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) = \mathrm{Index}_{D_8,\mathbb{Z}}^{\dim V+1} S(R_4^{\oplus j}) = \langle \partial_{3j,\mathbb{Z}}^{0,3j-1}(1 \otimes l)\rangle$$

where $l \in H^{3j-1}(S(R_4^{\oplus j}), \mathbb{Z})$ is a generator. The coefficient reduction morphism $j : \mathbb{Z} \to \mathbb{F}_2$ induces a morphism of Serre spectral sequences (Proposition 3.7. E. 3) associated with Borel construction of the sphere $S(R_4^{\oplus j})$. Thus,

$$j^*\left(\partial_{3j,\mathbb{Z}}^{0,3j-1}(1 \otimes l)\right) = \partial_{3j,\mathbb{F}_2}^{0,3j-1}\left(j^*(1 \otimes l)\right) \in H^{3j}(D_8, \mathbb{F}_2)$$

and according to the result of the previous section

$$j^*\left(\partial_{3j,\mathbb{Z}}^{0,3j-1}(1 \otimes l)\right) = y^j w^j.$$

Now, from the description of the map $j : H^*(D_8, \mathbb{Z}) \longrightarrow H^*(D_8, \mathbb{F}_2)$ in (39) follows the statement for j even.

6.2 The case when j is odd

The group D_8 acts non-trivially on the cohomology $H^*(S(R_4^{\oplus j}), \mathbb{Z})$. Precisely, the D_8-module $\mathcal{Z} = H^{3j-1}(S(R_4^{\oplus j}), \mathbb{Z})$ is a nontrivial D_8-module and for $z \in \mathcal{Z}$:

$$\varepsilon_1 \cdot z = z, \qquad \varepsilon_2 \cdot z = z, \qquad \sigma \cdot z = -z.$$

Then the E_2-term of the Serre spectral sequence associated to $\mathrm{E}D_8 \times_{D_8} S(R_4^{\oplus j})$ is not a tensor product and

$$E_2^{p,q} = H^p(D_8, H^q(S(R_4^{\oplus j}), \mathbb{Z})) = \begin{cases} H^p(D_8, \mathbb{Z}) & , q = 0 \\ H^p(D_8, \mathcal{Z}) & , q = 3j-1 \\ 0 & , q \neq 0, 3j-1 \end{cases}. \tag{49}$$

To compute the index in this case we have to study the $H^*(D_8, \mathbb{Z})$-module structure of $H^*(D_8, \mathcal{Z})$. Since the use of LHS-spectral sequence, like in the case of field coefficients (Proposition 7.4), can not be of significant help we apply the Bockstein spectral sequence associated with the following exact sequence of D_8-modules:

$$0 \to \mathcal{Z} \xrightarrow{\times 2} \mathcal{Z} \to \mathbb{F}_2 \to 0 \tag{50}$$

Proposition 6.1.

(A) $2 \cdot H^*(D_8, \mathcal{Z}) = 0$

(B) *$H^*(D_8, \mathcal{Z})$ is generated as a $H^*(D_8, \mathbb{Z})$-module by three elements ρ_1, ρ_2, ρ_3 of degree 1, 2, 3 such that*

$$\rho_1 \cdot \mathcal{Y} = 0,\ \rho_2 \cdot \mathcal{X} = 0,\ \rho_3 \cdot \mathcal{X} = 0$$

and

$$j(\rho_1) = x,\ j(\rho_2) = y^2,\ j(\rho_3) = yw$$

where j is the map induced by the map $\mathcal{Z} \to \mathbb{F}_2$ from the exact sequence (50).

Proof. The strategy of the proof is to consider four exact couples induced by the exact sequence (50):

$$\begin{tikzcd} H^*(D_8,\mathcal{Z}) \arrow[rr, "\times 2"] & & H^*(D_8,\mathcal{Z}) \arrow[dl, "j"] \\ & H^*(D_8,\mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd} \quad \begin{tikzcd} H^*(H_1,\mathcal{Z}) \arrow[rr, "\times 2"] & & H^*(H_1,\mathcal{Z}) \arrow[dl, "j"] \\ & H^*(H_1,\mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}$$

$$\begin{tikzcd} H^*(H_2,\mathcal{Z}) \arrow[rr, "\times 2"] & & H^*(H_2,\mathcal{Z}) \arrow[dl, "j"] \\ & H^*(H_2,\mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd} \quad \begin{tikzcd} H^*(K_4,\mathcal{Z}) \arrow[rr, "\times 2"] & & H^*(K_4,\mathcal{Z}) \arrow[dl, "j"] \\ & H^*(K_4,\mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}$$

and the corresponding morphisms induced by $\mathrm{res}^{D_8}_{H_1}$, $\mathrm{res}^{D_8}_{H_2}$ and $\mathrm{res}^{D_8}_{K_4}$. Our notation is as in the restriction diagram (26).

1. The module $\mathcal{Z}$ as a H_1-module is a trivial module. Therefore in the H_1 exact couple d_1 is the usual Bockstein homomorphism and so

$$d_1(a) = \mathrm{Sq}^1(a) = a^2, \qquad d_1(b) = \mathrm{Sq}^1(b) = b^2.$$

Thus from the restriction homomorphism $\mathrm{res}^{D_8}_{H_1}$ we have:

$$\begin{aligned} \mathrm{res}^{D_8}_{H_1}(d_1(1)) &= d_1\left(\mathrm{res}^{D_8}_{H_1}(1)\right) = d_1(1) = 0 &&\Rightarrow\ d_1(1) \in \ker\left(\mathrm{res}^{D_8}_{H_1}\right), \\ \mathrm{res}^{D_8}_{H_1}(d_1(x)) &= d_1\left(\mathrm{res}^{D_8}_{H_1}(x)\right) = d_1(0) = 0 &&\Rightarrow\ d_1(x) \in \ker\left(\mathrm{res}^{D_8}_{H_1}\right), \\ \mathrm{res}^{D_8}_{H_1}(d_1(y)) &= d_1\left(\mathrm{res}^{D_8}_{H_1}(y)\right) = d_1(b) = b^2 &&\Rightarrow\ d_1(y) \in y^2 + \ker\left(\mathrm{res}^{D_8}_{H_1}\right), \\ \mathrm{res}^{D_8}_{H_1}(d_1(w)) &= d_1\left(\mathrm{res}^{D_8}_{H_1}(w)\right) = ba(a+b) &&\Rightarrow\ d_1(w) \in yw + \ker\left(\mathrm{res}^{D_8}_{H_1}\right), \end{aligned} \tag{51}$$

2. The module $\mathcal{Z}$ as a H_2-module is a non-trivial module. The $H_2 \cong \mathbb{Z}_4$ exact couple unrolls into a long exact sequence [6, Proposition 6.1, page 71]

$$\begin{array}{cccccc} 0 & \longrightarrow & \begin{array}{c} H^0(\mathbb{Z}_4,\mathcal{Z}) \\ 0 \end{array} & \xrightarrow{\times 2} & \begin{array}{c} H^0(\mathbb{Z}_4,\mathcal{Z}) \\ 0 \end{array} & \longrightarrow & \begin{array}{c} H^0(\mathbb{Z}_4,\mathbb{F}_2) \\ \mathbb{F}_2\ \ ,\ 1 \end{array} \\ & \xrightarrow{\delta} & \begin{array}{c} H^1(\mathbb{Z}_4,\mathcal{Z}) \\ \mathbb{F}_2\ \ ,\ \lambda \end{array} & \xrightarrow{\times 2} & \begin{array}{c} H^1(\mathbb{Z}_4,\mathcal{Z}) \\ \mathbb{F}_2\ \ ,\ \lambda \end{array} & \longrightarrow & \begin{array}{c} H^1(\mathbb{Z}_4,\mathbb{F}_2) \\ \mathbb{F}_2\ \ ,\ e \end{array} \\ & \xrightarrow{\delta} & \begin{array}{c} H^2(\mathbb{Z}_4,\mathbb{Z}) \\ 0 \end{array} & \xrightarrow{\times 2} & \begin{array}{c} H^2(\mathbb{Z}_4,\mathbb{Z}) \\ 0 \end{array} & \longrightarrow & \begin{array}{c} H^2(\mathbb{Z}_4,\mathbb{F}_2) \\ \mathbb{F}_2\ \ ,\ u \end{array} \\ & \xrightarrow{\delta} & \begin{array}{c} H^3(\mathbb{Z}_4,\mathbb{Z}) \\ \mathbb{F}_2\ \ ,\ \lambda U \end{array} & \xrightarrow{\times 2} & \begin{array}{c} H^3(\mathbb{Z}_4,\mathbb{Z}) \\ \mathbb{F}_2\ \ ,\ \lambda U \end{array} & \longrightarrow & \begin{array}{c} H^3(\mathbb{Z}_4,\mathbb{F}_2) \\ \mathbb{F}_2\ \ ,\ eu \end{array} \\ & \xrightarrow{\delta} & \begin{array}{c} H^4(\mathbb{Z}_4,\mathbb{Z}) \\ 0 \end{array} & & \ldots & & \end{array}$$

Here we used facts that $H^i(\mathbb{Z}_4, \mathcal{Z}) = \left\{ \begin{array}{ll} \mathbb{F}_2 & , i \text{ odd} \\ 0 & , i \text{ even} \end{array} \right.$ and that multiplication by $U \in H^2(\mathbb{Z}_4, \mathbb{Z})$ in $H^*(\mathbb{Z}_4, \mathcal{Z})$ is an isomorphism [9, Section XII. 7. pages 250-253]. The long exact sequence describes the boundary operator:

$$\delta(1) = \lambda, \quad \delta(e) = 0, \quad \delta(u) = \lambda U$$

and consequently the first differential:

$$d_1(1) = e, \quad d_1(e) = 0, \quad d_1(u) = eu.$$

The restriction homomorphism $\mathrm{res}^{D_8}_{H_2}$ implies that:

$$\begin{array}{lcl} \mathrm{res}^{D_8}_{H_2}(d_1(1)) = d_1\left(\mathrm{res}^{D_8}_{H_1}(1)\right) = d_1(1) = e & \Rightarrow & d_1(1) \in x + \ker\left(\mathrm{res}^{D_8}_{H_2}\right), \\ \mathrm{res}^{D_8}_{H_1}(d_1(x)) = d_1\left(\mathrm{res}^{D_8}_{H_1}(x)\right) = d_1(e) = 0 & \Rightarrow & d_1(x) \in \ker\left(\mathrm{res}^{D_8}_{H_2}\right), \\ \mathrm{res}^{D_8}_{H_1}(d_1(y)) = d_1\left(\mathrm{res}^{D_8}_{H_1}(y)\right) = d_1(e) = 0 & \Rightarrow & d_1(y) \in \ker\left(\mathrm{res}^{D_8}_{H_2}\right), \\ \mathrm{res}^{D_8}_{H_1}(d_1(w)) = d_1\left(\mathrm{res}^{D_8}_{H_1}(w)\right) = d_1(u) = eu & \Rightarrow & d_1(w) \in yw + \ker\left(\mathrm{res}^{D_8}_{H_2}\right), \end{array} \tag{52}$$

3. The module $\mathcal{Z}$ as a K_4-module is a non-trivial module. Then the $K_4 \cong \mathbb{Z}_2$ exact couple unrolls into

$$\begin{array}{ccccccc} 0 & \to & \begin{array}{c} H^0(\mathbb{Z}_2,\mathcal{Z}) \\ 0 \end{array} & \xrightarrow{\times 2} & \begin{array}{c} H^0(\mathbb{Z}_2,\mathcal{Z}) \\ 0 \end{array} & \to & \begin{array}{c} H^0(\mathbb{Z}_2,\mathbb{F}_2) \\ \mathbb{F}_2 \ , \ 1 \end{array} \\ & \xrightarrow{\delta} & \begin{array}{c} H^1(\mathbb{Z}_2,\mathcal{Z}) \\ \mathbb{F}_2 \ , \ \varphi \end{array} & \xrightarrow{\times 2} & \begin{array}{c} H^1(\mathbb{Z}_2,\mathcal{Z}) \\ \mathbb{F}_2 \ , \ \varphi \end{array} & \to & \begin{array}{c} H^1(\mathbb{Z}_2,\mathbb{F}_2) \\ \mathbb{F}_2 \ , \ t_4 \end{array} \\ & \xrightarrow{\delta} & \begin{array}{c} H^2(\mathbb{Z}_2,\mathbb{Z}) \\ 0 \end{array} & \xrightarrow{\times 2} & \begin{array}{c} H^2(\mathbb{Z}_2,\mathbb{Z}) \\ 0 \end{array} & \to & \begin{array}{c} H^2(\mathbb{Z}_2,\mathbb{F}_2) \\ \mathbb{F}_2 \ , \ t_4^2 \end{array} \\ & \xrightarrow{\delta} & \begin{array}{c} H^3(\mathbb{Z}_2,\mathbb{Z}) \\ \mathbb{F}_2 \ , \ \varphi T \end{array} & \xrightarrow{\times 2} & \begin{array}{c} H^3(\mathbb{Z}_2,\mathbb{Z}) \\ \mathbb{F}_2 \ , \ \varphi T \end{array} & \to & \begin{array}{c} H^3(\mathbb{Z}_2,\mathbb{F}_2) \\ \mathbb{F}_2 \ , \ t_4^3 \end{array} \\ & \xrightarrow{\delta} & \begin{array}{c} H^4(\mathbb{Z}_2,\mathbb{Z}) \\ 0 \end{array} & & \dots & & \end{array} \tag{53}$$

Similarly, $H^i(\mathbb{Z}_2,\mathcal{Z}) = \begin{cases} \mathbb{F}_2 & , i \text{ odd} \\ 0 & , i \text{ even} \end{cases}$ and multiplication by $T \in H^2(\mathbb{Z}_2,\mathbb{Z})$ in $H^*(\mathbb{Z}_2,\mathcal{Z})$ is an isomorphism [9, Section XII. 7. pages 250-253]. Then

$$d_1(1) = t_4, \qquad d_1(t_4^{2i+1}) = 0, \qquad d_1(t_4^{2i}) = t_4^{2i+1}$$

for $i \geq 0$. This implies that

$$\mathrm{res}^{D_8}_{K_4}(d_1(w)) = d_1\left(\mathrm{res}^{D_8}_{H_1}(w)\right) = d_1(0) = 0 \tag{54}$$

and

$$\mathrm{res}^{D_8}_{K_4}(d_1(y)) = d_1\left(\mathrm{res}^{D_8}_{H_1}(y)\right) = d_1(0) = 0. \tag{55}$$

From (51), (52) and the restriction diagram (26) follows:

$$d_1(1) = x, \qquad d_1(x) = 0,$$

and

$$d_1(w) \in \{yw,\ yw + x^3\} \quad \text{and} \quad d_1(y) \in \{y^2, y^2 + x^2\}. \tag{56}$$

Since $\mathrm{res}^{D_8}_{K_4}(yw) = 0$, $\mathrm{res}^{D_8}_{K_4}(yw + x^3) = t_4^3 \neq 0$ and $\mathrm{res}^{D_8}_{K_4}(y^2) = 0$, $\mathrm{res}^{D_8}_{K_4}(y^2 + x^2) = t_4^2 \neq 0$, then the equations (54) and (55) resolve the final dilemmas (56). Thus $d_1(w) = yw$. According to [8, Remark 5.7.4, page 108] there are elements ρ_1, ρ_2, ρ_3 of degree 1, 2, 3 and of exponent 2 in $H^*(D_8,\mathcal{Z})$ satisfying property (B) of this proposition.

The property (A) follows from the properties of Bockstein spectral sequence and the fact that the derived couple of the D_8 exact couple is:

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ {}^{\delta}\nwarrow & & \swarrow \\ & 0 & \end{array}$$

where $\mathbb{F}_2$ appears in dimension 0. $\square$

Thus, the index is given by

$$\mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) = \langle \partial_{3j,\mathbb{Z}}^{1,3j-1}(\rho_1), \partial_{3j,\mathbb{Z}}^{2,3j-1}(\rho_2), \partial_{3j,\mathbb{Z}}^{3,3j-1}(\rho_3)\rangle.$$

The morphism J from spectral sequence (49) to spectral sequence (43) induced by the reduction homomorphism $\mathbb{Z} \to \mathbb{F}_2$ implies that:

$$\begin{array}{lllllllll} J(\partial_{3j,\mathbb{Z}}^{1,3j-1}(\rho_1)) & = & \partial_{3j,\mathbb{F}_2}^{1,3j-1}(j(\rho_1)) & = & \partial_{3j,\mathbb{F}_2}^{1,3j-1}(x) & = & 0 & & \\ J(\partial_{3j,\mathbb{Z}}^{2,3j-1}(\rho_2)) & = & \partial_{3j,\mathbb{F}_2}^{2,3j-1}(j(\rho_2)) & = & \partial_{3j,\mathbb{F}_2}^{2,3j-1}(y^2) & = & y^{j+2}w^j & = y^{j+1}w^{j-1}(y+x)w & \\ J(\partial_{3j,\mathbb{Z}}^{3,3j-1}(\rho_3)) & = & \partial_{3j,\mathbb{F}_2}^{3,3j-1}(j(\rho_3)) & = & \partial_{3j,\mathbb{F}_2}^{3,3j-1}(yw) & = & y^{j+1}w^{j+1} & & \end{array} \tag{57}$$

The sequence of D_8 inclusion maps

$$S(R_4^{\oplus(j-1)}) \subset S(R_4^{\oplus j}) \subset S(R_4^{\oplus(j+1)})$$

provides (Proposition 3.2) a sequence of inclusions:

$$\langle \mathcal{Y}^{\frac{j-1}{2}}\mathcal{W}^{\frac{j-1}{2}}\rangle = \mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus(j-1)}) \supseteq \mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) \supseteq \mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus(j+1)}) = \langle \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}}\rangle. \tag{58}$$

The relations (57), (58) and (39), along with Proposition 6.1 imply that for j odd:

$$\mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) = \langle \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M}, \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}}\rangle.$$

Remark 6.2. The index $\mathrm{Index}_{D_8,\mathbb{Z}} S(U_k \times R_4^{\oplus j})$ appearing in the join test maps scheme can now be computed. From Example 3.4 and the restriction diagram (42) it follows that

$$\mathrm{Index}_{D_8,\mathbb{Z}} S(U_k) = \mathrm{Index}_{D_8,\mathbb{Z}} D_8/H_1 = \ker\left(\mathrm{res}_{H_1}^{D_8} : H^*(D_8,\mathbb{Z}) \to H^*(H_1,\mathbb{Z})\right) = \langle \mathcal{X}\rangle.$$

The inclusions

$$\mathrm{Index}_{D_8,\mathbb{Z}} S(U_k \times R_4^{\oplus j}) \subseteq \mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) \quad \text{and} \quad \mathrm{Index}_{D_8,\mathbb{Z}} S(U_k \times R_4^{\oplus j}) \subseteq \mathrm{Index}_{D_8,\mathbb{Z}} S(U_k)$$

imply that

$$\mathrm{Index}_{D_8,\mathbb{Z}} S(U_k \times R_4^{\oplus j}) \subseteq \mathrm{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) \cap \mathrm{Index}_{D_8,\mathbb{Z}} S(U_k) = \{0\}.$$

Thus, as in the case of $\mathbb{F}_2$ coefficients, the Fadell–Husseini index theory with $\mathbb{Z}$ coefficients on the join CS/TM scheme does not lead to any non-trivial result.

7 $\mathbf{Index}_{D_8,\mathbb{F}_2}\mathbf{S}^d\times\mathbf{S}^d$

This section contains the proof of the equality

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S^d \times S^d = \langle \pi_{d+1}, \pi_{d+2}, w^{d+1}\rangle. \tag{59}$$

The index will be determined by the explicit computation of the Serre spectral sequence associated with the Borel construction

$$S^d \times S^d \to \mathrm{E}D_8 \times_{D_8} \left(S^d \times S^d\right) \to \mathrm{B}D_8.$$

The group D_8 acts nontrivially on the cohomology of the fibre, and therefore the spectral sequence has nontrivial local coefficients. The E_2-term is given by

$$\begin{array}{lll} E_2^{p,q} & = & H^p(\mathrm{B}D_8, \mathcal{H}^q(S^d \times S^d, \mathbb{F}_2)) = H^p(D_8, H^q(S^d \times S^d, \mathbb{F}_2)) \\ & = & \left\{ \begin{array}{ll} H^p(D_8, \mathbb{F}_2) & , q = 0, 2d \\ H^p(D_8, \mathbb{F}_2[D_8/H_1]) & , q = d \\ 0 & , q \neq 0, d, 2d \end{array} \right. . \end{array} \tag{60}$$

The nontriviality of local coefficients appears in at the d-th row of the spectral sequence.

In Section 7.4 there is a sketch of an alternative proof of the fact (59) suggested by a referee for an earlier, $\mathbb{F}_2$-coefficient, version of the paper.

7.1 The d-th row as an $H^*(D_8, \mathbb{F}_2)$-module

Since the spectral sequence is an $H^*(D_8, \mathbb{F}_2)$-module and the differentials are module maps we need to understand the $H^*(D_8, \mathbb{F}_2)$-module structure of the E_2-term. This can be done in at least two ways [25].

Proposition 7.1. $H^*(D_8, \mathbb{F}_2[D_8/H_1]) \cong_{\text{ring}} H^*(H_1, \mathbb{F}_2)$.

Proof. Here $H_1 = \langle \varepsilon_1, \varepsilon_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a maximal (normal) subgroup of index 2 in D_8.
Method 1: The statement follows from Shapiro's lemma [6, Proposition 6.2, page 73] and the fact that when $[G : H] < \infty$, then there is an isomorphism of G-modules $\text{Coind}_H^G M \cong \text{Ind}_H^G M$.
Method 2: There is an exact sequence of groups

$$1 \to H_1 \to D_8 \to D_8/H_1 \to 1.$$

The associated LHS spectral sequence [1, Corollary 1.2, page 116] has the E_2-term:

$$\begin{aligned} A_2^{p,q} &= H^p(D_8/H_1, H^q(H_1, \mathbb{F}_2[D_8/H_1])) \\ &\cong H^p(\mathbb{Z}_2, H^q((\mathbb{Z}_2)^2, \mathbb{F}_2 \oplus \mathbb{F}_2)) \\ &\cong H^p(\mathbb{Z}_2; H^q((\mathbb{Z}_2)^2, \mathbb{F}_2) \oplus H^q((\mathbb{Z}_2)^2, \mathbb{F}_2)) \end{aligned}$$

The action of the group $D_8/H_1 \cong \mathbb{Z}_2$ on the sum is given by the conjugation action of G on the pair $(H_1, H^q(H_1, \mathbb{F}_2[D_8/H_1]))$ [6, Corollary 8.4, page 80]. Since $\mathbb{F}_2[\mathbb{Z}_2]$ is a free $\mathbb{Z}_2$-module

$$H^0(\mathbb{Z}_2; \mathbb{F}_2[\mathbb{Z}_2]) = (\mathbb{F}_2[\mathbb{Z}_2])^{\mathbb{Z}_2} = \mathbb{F}_2$$

and $H^p(\mathbb{Z}_2; \mathbb{F}_2[\mathbb{Z}_2]) = 0$ for $p > 0$. Thus

$$\begin{aligned} A_2^{p,q} &\cong H^p(D_8/H_1; H^q((\mathbb{Z}_2)^2, \mathbb{F}_2) \oplus H^q((\mathbb{Z}_2)^2, \mathbb{F}_2)) \\ &\cong H^p(D_8/H_1; \mathbb{F}_2[\mathbb{Z}_2]^{q+1}) \\ &\cong H^p(D_8/H_1; \mathbb{F}_2[\mathbb{Z}_2])^{q+1} \cong \begin{cases} \left(H^p(\mathbb{Z}_2; \mathbb{F}_2[\mathbb{Z}_2])^{q+1}\right)^{\mathbb{Z}_2} \cong \mathbb{F}_2^{q+1} & , p = 0 \\ 0 & , p > 0 \end{cases}. \end{aligned}$$

Thus the E_2-term has the shape as in Figure 6 (concentrated in the 0-column) and collapses. $\square$

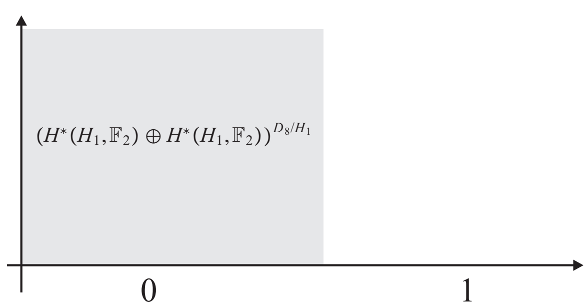

Figure 6: The A_2-term of the LHS spectral sequence

The first information about the $H^*(D_8, \mathbb{F}_2)$-module structure on $H^*(D_8, \mathbb{F}_2[D_8/H_1])$, as well as the method for revealing the complete structure, are coming from the following proposition.

Proposition 7.2. *We have $x \cdot H^*(D_8, \mathbb{F}_2[D_8/H_1]) = 0$ for the element $x \in H^1(D_8, \mathbb{F}_2)$ that is characterized by $\text{res}_{H_1}^{D_8}(x) = 0$.*

Proof.
Method 1: The isomorphism $H^*(D_8, \mathbb{F}_2[D_8/H_1]) \cong_{\text{ring}} H^*(H_1, \mathbb{F}_2)$ induced by Shapiro's lemma [6, Proposition 6.2, page 73] carries the $H^*(D_8, \mathbb{F}_2)$-module structure to $H^*(H_1, \mathbb{F}_2)$ via the restriction homomorphism $\text{res}_{H_1}^{D_8} : H^*(D_8, \mathbb{F}_2) \to H^*(H_1, \mathbb{F}_2)$. In this way the complete $H^*(D_8, \mathbb{F}_2)$-module structure is given on $H^*(D_8, \mathbb{F}_2[D_8/H_1])$. Particularly, since $\text{res}_{H_1}^{D_8}(x) = 0$, the proposition is proved.

Method 2: The exact sequence of groups

$$1 \to H_1 \to D_8 \to D_8/H_1 \to 1$$

induces two LHS spectral sequences

$$A_2^{p,q} = H^p\left(D_8/H_1, H^q\left(H_1, \mathbb{F}_2[D_8/H_1]\right)\right) \implies H^{p+q}(D_8, \mathbb{F}_2[D_8/H_1]), \tag{61}$$

$$B_2^{p,q} = H^p\left(D_8/H_1, H^q\left(H_1, \mathbb{F}_2\right)\right) \implies H^{p+q}(D_8, \mathbb{F}_2). \tag{62}$$

The spectral sequence (62) acts on the spectral sequence (61)

$$B_t^{r,s} \times A_t^{u,v} \to A_t^{u+r,v+s}$$

In the ∞-term this action becomes an action of $H^*(D_8, \mathbb{F}_2)$ on $H^*(D_8, \mathbb{F}_2[D_8/H_1])$. Since we already discussed both spectral sequences we know that

$$A_2^{p,q} = A_\infty^{p,q} \quad \text{and} \quad B_2^{p,q} = B_\infty^{p,q}.$$

From Figures 3 and 6 it is apparent that $x \in B_2^{1,0} = B_\infty^{1,0}$ acts by

$$x \cdot A_2^{p,q} = 0$$

for every p, q. $\square$

Corollary 7.3. $\mathrm{Index}_{D_8,,\mathbb{F}_2}^{d+2} S^d \times S^d = \mathrm{im}\big(\partial_{d+1} : E_{d+1}^{*,d} \to E_{d+1}^{*+d+1,0}\big) \subseteq y \cdot H^*(D_8\mathbb{F}_2)$.

Proof. Let $\alpha \in E_{d+1}^{*,d}$ and $\partial_{d+1}(\alpha) \notin y \cdot H^*(D_8\mathbb{F}_2)$. Then $x \cdot \partial_{d+1}(\alpha) \neq 0$. Since ∂_{d+1} is a $H^*(D_8\mathbb{F}_2)$-module map and x acts trivially on $H^*(D_8, \mathbb{F}_2[D_8/H_1])$, as indicated by Proposition 7.2, there is a contradiction

$$0 = \partial_{d+1}(x \cdot \alpha) = x \cdot \partial_{d+1}(\alpha) \neq 0.$$

$\square$

Proposition 7.4. *$H^*(D_8, \mathbb{F}_2[D_8/H_1])$ is generated as an $H^*(D_8, \mathbb{F}_2)$-module by*

$$H^0(D_8, \mathbb{F}_2[D_8/H_1]) \quad \textit{and} \quad H^1(D_8, \mathbb{F}_2[D_8/H_1]).$$

Proof.
Method 1: We already observed that Shapiro's lemma $H^*(D_8, \mathbb{F}_2[D_8/H_1]) \cong_{\mathrm{ring}} H^*(H_1, \mathbb{F}_2)$ carries the $H^*(D_8, \mathbb{F}_2)$-module structure to $H^*(H_1, \mathbb{F}_2)$ via the restriction homomorphism $\mathrm{res}_{H_1}^{D_8} : H^*(D_8, \mathbb{F}_2) \to H^*(H_1, \mathbb{F}_2)$. Thus $H^*(H_1, \mathbb{F}_2)$ as an $H^*(D_8, \mathbb{F}_2)$-module is generated by $1 \in H^0(D_8, \mathbb{F}_2)$ together with $a \in H^1(D_8, \mathbb{F}_2)$.
Method 2: There is the exact sequence of D_8-modules

$$0 \to \mathbb{F}_2 \to \mathbb{F}_2[D_8/H_1] \to \mathbb{F}_2 \to 0, \tag{63}$$

where the left and right modules $\mathbb{F}_2$ are trivial D_8-modules. The first map is a diagonal inclusion while the second on is a quotient map. The sequence (63) induces a long exact sequence on group cohomology [6, Proposition 6.1, page 71],

$$\begin{aligned} 0 \to \quad & H^0\left(D_8, \mathbb{F}_2\right) \xrightarrow{i_0} H^0\left(D_8, \mathbb{F}_2[D_8/H_1]\right) \xrightarrow{q_o} H^0\left(D_8, \mathbb{F}_2\right) \xrightarrow{\delta_0} \\ & H^1\left(D_8, \mathbb{F}_2\right) \xrightarrow{i_1} H^1\left(D_8, \mathbb{F}_2[D_8/H_1]\right) \xrightarrow{q_1} H^1\left(D_8, \mathbb{F}_2\right) \xrightarrow{\delta_1} \quad \ldots \end{aligned} \tag{64}$$

From the exact sequence (64), compatibility of the cup product [6, page 110, (3.3)] and Proposition 7.2 one can deduce that $\delta_0(1) = x$. Then by chasing along sequence (64) with compatibility of the cup product [6, page 110,(3.3)] as a tool it can be proved that $H^*(D_8, \mathbb{F}_2[D_8/H_1])$ is generated as a $H^*(D_8, \mathbb{F}_2)$-module by $I = i_0(1)$ and $A \in q_1^{-1}(\{y\})$. $\square$

7.2 $\text{Index}^{d+2}_{D_8,\mathbb{F}_2} S^d \times S^d = \langle \pi_{d+1}, \pi_{d+2} \rangle$

The index by definition is

$$\begin{aligned}\text{Index}^{d+2}_{D_8,\mathbb{F}_2} S^d \times S^d &= \text{im}\big(\partial_{d+1} : E^{*,d}_{d+1} \to E^{*+d+1,0}_{d+1}\big)\\ &= \text{im}\,\big(\partial_{d+1} : H^*\left(D_8, \mathbb{F}_2[D_8/H_1]\right) \to H^{*+d+1}(D_8, \mathbb{F}_2)\big).\end{aligned}$$

From Proposition 7.4 this image is generated as a module by the ∂_{d+1}-images of $H^0\left(D_8, \mathbb{F}_2[D_8/H_1]\right)$ and of $H^1\left(D_8, \mathbb{F}_2[D_8/H_1]\right)$. The ∂_{d+1} image is computed by applying restriction properties given in Proposition 3.5 to the subgroup H_1. With the identification of $H^*\left(D_8, \mathbb{F}_2[D_8/H_1]\right)$ given by Shapiro's lemma the morphism of spectral sequences of Borel constructions induced by restriction is specified in Figure 7. Also,

$$\text{Index}^{d+2}_{D_8,\mathbb{F}_2} S^d \times S^d = \langle \partial^{D_8}_{d+1}(1), \partial^{D_8}_{d+1}(a), \partial^{D_8}_{d+1}(b), \partial^{D_8}_{d+1}(a+b) \rangle.$$

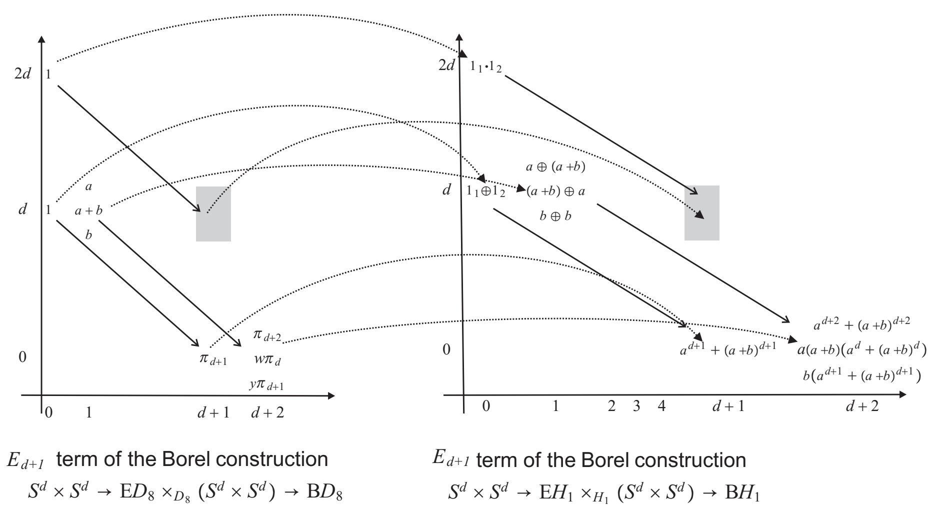

Figure 7: The morphism of spectral sequences

To simplify notation let $\rho_d := a^d + (a+b)^{d+1}$. Then from

$$1 \quad \overset{\text{res}^{D_8}_{H_1}}{\longmapsto} \quad 1_1 \oplus 1_2 \quad \overset{\partial^{H_1}_{d+1}}{\longmapsto} \quad \rho_{d+1}$$

$$\{a, a+b, b\} \quad \overset{\text{res}^{D_8}_{H_1}}{\longmapsto} \quad \left\{\begin{matrix} a \oplus (a+b) \\ (a+b) \oplus a \\ b \oplus b \end{matrix}\right\} \quad \overset{\partial^{H_1}_{d+1}}{\longmapsto} \quad \{\rho_{d+2}, a(a+b)\rho_d, b\rho_{d+1}\}$$

it follows that

$$\text{res}^{D_8}_{H_1}\big(\{\partial^{D_8}_{d+1}(1),\ \partial^{D_8}_{d+1}(a),\ \partial^{D_8}_{d+1}(b)\ , \partial^{D_8}_{d+1}(a+b)\}\big) = \{\rho_{d+2},\ a(a+b)\rho_d,\ b\rho_{d+1}\}.$$

The formula

$$\begin{aligned}\rho_{d+2} &= a^{d+2} + (a+b)^{d+2}\\ &= (a+a+b)\Big(\rho_{d+1} + a(a+b)\sum_{i=0}^{d-1} a^i (a+b)^{d-1-i}\Big)\\ &= b\rho_{d+1} + a(a+b)(a+a+b)\sum_{i=0}^{d-1} a^i (a+b)^{d-1-i}\\ &= b\rho_{d+1} + a(a+b)\rho_d\end{aligned}$$

together with Remark 1.3 and the knowledge of the restriction $\mathrm{res}^{D_8}_{H_1}$ implies that

$$\mathrm{res}^{D_8}_{H_1}(\pi_d) = \rho_d.$$

Therefore, there exist $x\alpha, x\beta, x\gamma, x\delta \in \ker(\mathrm{res}^{D_8}_{H_1})$ such that

$$\partial^{D_8}_{d+1}(1) = \pi_{d+1} + x\alpha$$

and

$$\left\{\partial^{D_8}_{d+1}(a),\ \partial^{D_8}_{d+1}(b),\ \partial^{D_8}_{d+1}(a+b)\right\} = \left\{\pi_{d+2} + x\beta,\ y\pi_{d+1} + x\gamma,\ w\pi_d + x\delta\right\}.$$

Since y divides π_d, Proposition 7.2 implies that $\alpha = \beta = \gamma = \delta = 0$, and

$$\begin{aligned}
\mathrm{Index}^{d+2}_{D_8} S^d \times S^d &= \langle \partial^{D_8}_{d+1}(1), \partial^{D_8}_{d+1}(a), \partial^{D_8}_{d+1}(b), \partial^{D_8}_{d+1}(a+b)\rangle \\
&= \langle \pi_{d+1}, \pi_{d+2},\ y\pi_{d+1},\ w\pi_d\rangle \\
&= \langle \pi_{d+1}, \pi_{d+2}\rangle.
\end{aligned}$$

Remark 7.5. The property that the concretely described homomorphism

$$\mathrm{res}^{D_8}_{H_1} : H^*(D_8, \mathbb{F}_2[D_8/H_1]) \to H^*(H_1, \mathbb{F}_2[D_8/H_1])$$

is injective holds more generally [11, Lemma on page 187].

7.3 $\mathrm{Index}_{D_8,\mathbb{F}_2} S^d \times S^d = \langle \pi_{d+1}, \pi_{d+2}, w^{d+1}\rangle$

In the previous section we described the differential $\partial^{D_8}_{d+1}$ of the Serre spectral sequence associated with the Borel construction

$$S^d \times S^d \to \mathrm{E}D_8 \times_{D_8} \left(S^d \times S^d\right) \to \mathrm{B}D_8.$$

The only remaining, possibly non-trivial, differential is $\partial^{D_8}_{2d+1}$.

The following proposition describing $E^{*,2d}_{2d+1}$ can be obtained from the Figure 7.

Proposition 7.6. $E^{*,2d}_{2d+1} = \ker\left(\partial^{D_8}_{d+1} : E^{*,2d}_{d+1} \to E^{*+d+1,d}_{d+1}\right) = x \cdot H^*(D_8, \mathbb{F}_2)$

Proof. The restriction property from Proposition 3.5(D), applied on the element $1 \in E^{0,2d}_{d+1} = H^*(D_8, \mathbb{F}_2)$ implies that $\partial^{D_8}_{d+1}(1) \neq 0$. Proposition 7.2, together with the fact that multiplication by y and w in $H^*(D_8, \mathbb{F}_2[D_8/H_1])$ is injective, implies that $\ker\left(\partial^{D_8}_{d+1} : E^{*,2d}_{d+1} \to E^{*+d+1,d}_{d+1}\right) = xH^*(D_8, \mathbb{F}_2)$. □

The description of the differential $\partial^{D_8}_{2d+1} : E^{*,2d}_{2d+1} \to E^{*+2d+1,0}_{2d+1}$ comes in an indirect way. There is a D_8-equivariant map

$$S^d \times S^d \to S^d * S^d \approx S((V_{+-} \oplus V_{-+})^{\oplus(d+1)})$$

given as inclusion of the diagonal of a product into a join. The result of the Section 5.1 and the basic property of the index (Proposition 3.2) imply that

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S^d \times S^d \supseteq \mathrm{Index}_{D_8,\mathbb{F}_2} S((V_{+-} \oplus V_{-+})^{\oplus(d+1)}) = \langle w^{d+1}\rangle$$

Thus $w^{d+1} \in \mathrm{Index}_{D_8,\mathbb{F}_2} S^d \times S^d$. Since by Corollary 7.3 $w^{d+1} \notin \mathrm{Index}^{d+1}_{D_8,\mathbb{F}_2} S^d \times S^d$ it follows that

$$w^{d+1} \in \mathrm{im}\left(\partial^{D_8}_{2d+1} : E^{1,2d}_{2d+1} \to E^{2d+2,0}_{2d+1}\right).$$

But the only element in $E^{1,2d}_{2d+1}$ is x, therefore

$$\partial^{D_8}_{2d+1}(x) = w^{d+1}.$$

This concludes the proof of equation (59).

7.4 An alternative proof, sketch

The objective of our index calculation is to find the kernel of the map (cf. Section 3):

$$H^*(\mathrm{E}D_8 \times_{D_8} \left(S^d \times S^d\right), \mathbb{F}_2) = H^*_{D_8}(S^d \times S^d, \mathbb{F}_2) \leftarrow H^*_{D_8}(pt, \mathbb{F}_2) = H^*(\mathrm{E}D_8 \times_{D_8} pt, \mathbb{F}_2). \tag{65}$$

This map is induced by the map of spaces

$$\mathrm{E}D_8 \times_{D_8} (S^d \times S^d) \rightarrow \mathrm{E}D_8 \times_{D_8} pt. \tag{66}$$

From the definition of the product $\times_{D_8}$ the map (66) is induced by $\mathrm{E}D_8 \times (S^d \times S^d) \rightarrow \mathrm{E}D_8 \times pt$, i.e. by $(S^d \times S^d) \rightarrow pt$. The map (66), again by definition of product $\times_{D_8}$ is

$$\left(\mathrm{E}D_8 \times (S^d \times S^d)\right)/D_8 \rightarrow (\mathrm{E}D_8 \times pt)/D_8. \tag{67}$$

Let $S_2 \cong \mathbb{Z}_2$ denotes the quotient group D_8/H_1. There is a natural homeomorphisms [18, Proposition 1.59, page 40]

$$\left(\left(\mathrm{E}D_8 \times (S^d \times S^d)\right)/H_1\right)/S_2 \rightarrow ((\mathrm{E}D_8 \times pt)/H_1)/S_2 \tag{68}$$

which is induced by the map

$$\left(\mathrm{E}D_8 \times (S^d \times S^d)\right)/H_1 \rightarrow (\mathrm{E}D_8 \times pt)/H_1 \tag{69}$$

Since $\mathrm{E}D_8$ is also a model for $\mathrm{E}H_1$, the map (69) is a projection map in the Borel construction of $S^d \times S^d$ with respect to the group H_1:

$$\begin{array}{ccc} S^d \times S^d & \rightarrow & \left(\mathrm{E}D_8 \times (S^d \times S^d)\right)/H_1 \\ & & \downarrow \\ & & \mathrm{B}H_1 \end{array} \tag{70}$$

The group D_8 acts freely on $\mathrm{E}D_8 \times (S^d \times S^d)$ and on $\mathrm{E}D_8 \times pt$. Therefore S_2 action on $\left(\mathrm{E}D_8 \times (S^d \times S^d)\right)/H_1$ and $(\mathrm{E}D_8 \times pt)/H_1$ is also free. There are natural homotopy equivalences

$$\left(\left(\mathrm{E}D_8 \times (S^d \times S^d)\right)/H_1\right)/S_2 \simeq \mathrm{E}S_2 \times_{S_2} \left(\left(\mathrm{E}D_8 \times (S^d \times S^d)\right)/H_1\right),$$

$$((\mathrm{E}D_8 \times pt)/H_1)/S_2 \simeq \mathrm{E}S_2 \times_{S_2} ((\mathrm{E}D_8 \times pt)/H_1)$$

which transform the map (68) into a map of Borel constructions

$$\mathrm{E}S_2 \times_{S_2} \left(\left(\mathrm{E}D_8 \times (S^d \times S^d)\right)/H_1\right) \rightarrow \mathrm{E}S_2 \times_{S_2} ((\mathrm{E}D_8 \times pt)/H_1) \tag{71}$$

induced by the map (69) on the fibers.
The map between Borel constructions (71) induces a map of associated Serre spectral sequences which in the E_2-term looks like

$$\mathcal{E}_2^{p,q} = H^p(S_2, H^q((\mathrm{E}D_8 \times \left(S^d \times S^d\right))/H_1, \mathbb{F}_2)) \leftarrow H^p(S_2, H^q((\mathrm{E}D_8 \times pt)/H_1, \mathbb{F}_2)) = \mathcal{H}_2^{p,q}. \tag{72}$$

The spectral sequence $\mathcal{H}_2^{p,q}$ is the one studied in section 4.2. It converges to $H^*(D_8, \mathbb{F}_2)$ and $\mathcal{H}_2^{p,q} = \mathcal{H}_\infty^{p,q}$.

Lemma 7.7. $\mathcal{E}_2^{p,q} = \mathcal{E}_\infty^{p,q}$

Proof. The action of H_1 on $S^d \times S^d$ is free. Therefore

$$\left(\mathrm{E}D_8 \times \left(S^d \times S^d\right)\right)/H_1 \simeq \left(S^d \times S^d\right)/H_1 = \mathbb{R}P^d \times \mathbb{R}P^d \tag{73}$$

where the induced action of S_2 from $\left(\mathrm{E}D_8 \times \left(S^d \times S^d\right)\right)/H_1$ onto $\mathbb{R}P^d \times \mathbb{R}P^d$ interchanges the copies of $\mathbb{R}P^d \times \mathbb{R}P^d$. The S_2-homotopy equivalence (73) induces an isomorphism of induced Serre spectral sequences of Borel constructions

$$\mathcal{E}_2^{p,q} = H^p(S_2, H^q(\left(\mathrm{E}D_8 \times \left(S^d \times S^d\right)\right)/H_1, \mathbb{F}_2)) \cong H^p(S_2, H^q(\mathbb{R}P^d \times \mathbb{R}P^d, \mathbb{F}_2)) = \mathcal{G}_2^{p,q}.$$

Since for the spectral sequence $\mathcal{G}_2^{p,q}$, by [1, Theorem 1.7, page 118], we know that $\mathcal{G}_2^{p,q} = \mathcal{G}_\infty^{p,q}$, the same must hold for the spectral sequence $\mathcal{E}_*^{*,*}$. □

We obtained the following presentation of the map (65) and the related map of the fibers (69).

Proposition 7.8.
(A) *The map* $H^*_{D_8}(pt,\mathbb{F}_2) \rightarrow H^*_{D_8}(S^d\times S^d,\mathbb{F}_2)$ *can be seen as a map of spectral sequences of* S_2*-Borel constructions*

$$\mathcal{H}_2^{p,q} = H^p(S_2, H^q((\mathrm{E}D_8\times pt)/H_1,\mathbb{F}_2)) \rightarrow H^p(S_2, H^q((\mathrm{E}D_8\times \left(S^d\times S^d\right))/H_1,\mathbb{F}_2)) = \mathcal{E}_2^{p,q} \quad (74)$$

which is induced by the map on fibers $\left(\mathrm{E}D_8\times (S^d\times S^d)\right)/H_1 \rightarrow (\mathrm{E}D_8\times pt)/H_1$.

(B) *The map on the fibers is the projection map in the* H_1*-Borel construction*

$$S^d\times S^d \quad\rightarrow\quad \left(\mathrm{E}D_8\times (S^d\times S^d)\right)/H_1 \quad\rightarrow\quad \mathrm{B}H_1\ .$$

It is completely in $\mathbb{F}_2$ *cohomology determined by its kernel:*

$$\ker\left(H^*(H_1,\mathbb{F}_2)\rightarrow H^*(\left(\mathrm{E}D_8\times (S^d\times S^d)\right)/H_1,\mathbb{F}_2)\right) = \mathrm{Index}_{H_1,\mathbb{F}_2}S^d\times S^d = \langle a^{d+1},(a+b)^{d+1}\rangle.$$

The $\mathcal{E}_2^{p,q}=\mathcal{E}_\infty^{p,q}$ and $\mathcal{H}_2^{p,q}=\mathcal{H}_\infty^{p,q}$ are described by [1, Lemma 1.4, page 117]. Therefore, $\mathrm{Index}_{D_8,\mathbb{F}_2}S^d\times S^d$ or the kernel of the map of spectral sequences (74) is completely determined by the kernel of the map of S_2-invariants

$$\begin{array}{ccc} \mathbb{F}_2[a,a+b]^{S_2} & \rightarrow & \left(\mathbb{F}_2[a,a+b]/\langle a^{d+1},(a+b)^{d+1}\rangle\right)^{S_2} \\ \shortparallel & & \shortparallel \\ H^*(H_1,\mathbb{F}_2)^{S_2} & \rightarrow & H^*(\left(\mathrm{E}D_8\times (S^d\times S^d)\right)/H_1,\mathbb{F}_2)^{S_2} \end{array} \quad (75)$$

where S_2 action is given by $a\longmapsto a+b$. The equation (59)

$$\mathrm{Index}_{D_8,\mathbb{F}_2}S^d\times S^d = \langle \pi_{d+1},\pi_{d+2},w^{d+1}\rangle.$$

is a consequence of the previous discussion, identification of elements (22) in the spectral sequence (21) and the following proposition about symmetric polynomials.

Proposition 7.9.
(A) *A symmetric polynomial* $\sum a^{i_k}(a+b)^{j_k}\in\mathbb{F}_2[a,a+b]^{S_2}$ *is in the kernel of the map* (75) *if and only if for every monomial*

$$a^{d+1}\mid a^{i_k}(a+b)^{j_k}\ \text{or}\ (a+b)^{d+1}\mid a^{i_k}(a+b)^{j_k}.$$

(B) *The kernel of the map* (75), *as an ideal in* $\mathbb{F}_2[a,a+b]^{S_2}$ *is generated by*

$$a^{d+1}+(a+b)^{d+1},\qquad a^{d+2}+(a+b)^{d+2},\qquad a^{d+1}(a+b)^{d+1}.$$

The presented approach with all advantages has two disadvantages:
(1) The carrier of the combinatorial lower bound for the mass partition problem, the partial index $\mathrm{Index}^{d+2}_{D_8,\mathbb{F}_2}S^d\times S^d$, can not be obtained without extra effort.
(2) It can not be used for computation of the index $\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}}S^d\times S^d$; the spectral sequence $\mathcal{H}_2^{p,q}$, if considered with $\mathbb{Z}$ coefficients, is the sequence (34) whose ∞-term has a ring structure that different from $H^*(D_8,\mathbb{Z})$.

These were our reasons for presenting this idea just as a sketch.

8 $\mathbf{Index}_{D_8,\mathbb{Z}}\mathbf{S}^d\times\mathbf{S}^d$

Let $\Pi_0=0$, $\Pi_1=\mathcal{Y}$ and $\Pi_{n+2}=\mathcal{Y}\Pi_{n+1}+\mathcal{W}\Pi_n$, for $n\geq 0$, be a sequence of polynomials in $H^*(D_8,\mathbb{Z})$. This section contains the proof of the equality

$$\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}}S^d\times S^d = \begin{cases} \langle \Pi_{\frac{d+2}{2}},\Pi_{\frac{d+4}{2}},\mathcal{M}\Pi_{\frac{d}{2}}\rangle & ,\ \text{for } d \text{ even} \\ \langle \Pi_{\frac{d+1}{2}},\Pi_{\frac{d+3}{2}}\rangle & ,\ \text{for } d \text{ odd} \end{cases} \quad (76)$$

The index is determined by the explicit computation of the E_{d+2}-term of the Serre spectral sequence associated with the Borel construction

$$S^d \times S^d \to \mathrm{E}D_8 \times_{D_8} \left(S^d \times S^d\right) \to \mathrm{B}D_8.$$

Like in the previous section, the group D_8 acts nontrivially on the cohomology of the fibre and thus the coefficients in the spectral sequence are local. The E_2-term is given by

$$\begin{aligned} E_2^{p,q} &= H^p(\mathrm{B}D_8, \mathcal{H}^q(S^d \times S^d, \mathbb{Z})) = H^p(D_8, H^q(S^d \times S^d, \mathbb{Z})) \\ &= \begin{cases} H^p(D_8, \mathbb{Z}) & , q = 0, 2d \\ H^p(D_8, H^d(S^d \times S^d, \mathbb{Z})) & , q = d \\ 0 & , q \neq 0, d, 2d \end{cases} . \end{aligned} \tag{77}$$

The local coefficients are nontrivial in the d-th row of the spectral sequence.

8.1 The d-th row as an $H^*(D_8, \mathbb{Z})$-module

The D_8-module $M := H^d(S^d \times S^d, \mathbb{Z})$, as an abelian group, is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Since the action of D_8 on M depends on d we distinguish two cases.

8.1.1 The case when d is odd

The action on M is given by

$$\varepsilon_1 \cdot (x, y) = (x, y), \qquad \varepsilon_2 \cdot (x, y) = (x, y), \qquad \sigma \cdot (x, y) = (y, x).$$

Thus, there is and isomorphism of D_8-modules $M \cong \mathbb{Z}[D_8/H_1]$. The situation resembles the one in Section 7.1, and therefore following propositions hold.

Proposition 8.1. *$H^*(D_8, \mathbb{Z}[D_8/H_1]) \cong_{\text{ring}} H^*(H_1, \mathbb{Z})$.*

Proof. The claim follows from Shapiro's lemma [6, Proposition 6.2, page 73] and the fact that when $[G : H] < \infty$. There is an isomorphism of G-modules $\mathrm{Coind}_H^G M \cong \mathrm{Ind}_H^G M$. □

Proposition 8.2. *Let $\mathcal{T} \in H^*(D_8, \mathbb{Z})$ and $P \in H^*(H_1, \mathbb{Z}) \cong H^*(D_8, \mathbb{Z}[D_8/H_1])$.*

(A) *The action of $H^*(D_8, \mathbb{Z})$ on $H^*(D_8, \mathbb{Z}[D_8/H_1])$ is given by*

$$\mathcal{T} \cdot P := \mathrm{res}_{H_1}^{D_8}(\mathcal{T}) \cdot P.$$

Here P on the right hand side is an element of $H^*(H_1, \mathbb{Z})$ and on the left hand side its isomorphic image along the isomorphism from the previous proposition. Particularly, $\mathcal{X} \cdot H^*(D_8, \mathbb{Z}[D_8/H_1]) = 0$.

(B) *$H^*(D_8, \mathbb{Z})$-module $H^*(D_8, \mathbb{Z}[D_8/H_1])$ is generated by two elements*

$$1, \alpha \in H^*(H_1, \mathbb{Z}) \cong H^*(D_8, \mathbb{Z}[D_8/H_1])$$

of degree 0 and 2.

(C) *The map $H^*(D_8, \mathbb{Z}[D_8/H_1]) \to H^*(D_8, \mathbb{F}_2[D_8/H_1])$, induced by the coefficient map $\mathbb{Z} \to \mathbb{F}_2$, is given by $1, \alpha \longmapsto 1, a^2$*

Proof. The isomorphism $H^*(D_8, \mathbb{Z}[D_8/H_1]) \cong_{\text{ring}} H^*(H_1, \mathbb{Z})$ induced by Shapiro's lemma [6, Proposition 6.2, page 73] carries the $H^*(D_8, \mathbb{Z})$-module structure to $H^*(H_1, \mathbb{Z})$ via $\mathrm{res}_{H_1}^{D_8} : H^*(D_8, \mathbb{Z}) \to H^*(H_1, \mathbb{Z})$. In this way the complete $H^*(D_8, \mathbb{Z})$-module structure is given on $H^*(D_8, \mathbb{Z}[D_8/H_1])$. The claim (B) follows from the restriction diagram (42). The morphism of restriction diagrams J, induced by the coefficient reduction homomorphism $j : \mathbb{Z} \to \mathbb{F}_2$, implies the last statement. □

8.1.2 The case when d is even

The action on M is given by

$$\varepsilon_1 \cdot (x, y) = (-x, y), \qquad \varepsilon_2 \cdot (x, y) = (x, -y), \qquad \sigma \cdot (x, y) = (y, x).$$

In this case we are forced to analyze the Bockstein spectral sequence associated with the exact sequence of D_8-modules

$$0 \to M \xrightarrow{\times 2} M \to \mathbb{F}_2[D_8/H_1] \to 0, \tag{78}$$

i.e. with the exact couple

$$\begin{tikzcd} H^*(D_8, M) \arrow[rr, "\times 2"] & & H^*(D_8, M) \arrow[dl, "j"] \\ & H^*(D_8, \mathbb{F}_2[D_8/H_1]) \arrow[ul, "\delta"] & \end{tikzcd} \tag{79}$$

First we study the Bockstein spectral sequence

$$\begin{tikzcd} H^*(H_1, M) \arrow[rr, "\times 2"] & & H^*(H_1, M) \arrow[dl, "j"] \\ & H^*(H_1, \mathbb{F}_2[D_8/H_1]) \arrow[ul, "\delta"] & \end{tikzcd} . \tag{80}$$

Like in the Section 7.2, we have that $H^*(H_1, \mathbb{F}_2[D_8/H_1]) = \mathbb{F}_2[a, a+b] \oplus \mathbb{F}_2[a, a+b]$. The module M as an H_1-module can be decomposed into sum of two H_1-modules Z_1 and Z_2. The modules $Z_1 \cong_{Ab} \mathbb{Z}$ and $Z_2 \cong_{Ab} \mathbb{Z}$ are given by

$$\varepsilon_1 \cdot x = -x, \ \varepsilon_2 \cdot x = x \text{ and } \varepsilon_1 \cdot y = y, \ \varepsilon_2 \cdot y = -y$$

where $x \in Z_1$ and $y \in Z_2$. This decomposition also induces a decomposition of H_1-modules $\mathbb{F}_2[D_8/H_1] \cong \mathbb{F}_2 \oplus \mathbb{F}_2$. Thus, the exact couple (80) decomposes into the direct sum of two exact couples

$$\begin{tikzcd} H^*(H_1, Z_1) \arrow[rr, "\times 2"] & & H^*(H_1, Z_1) \arrow[dl, "j"] \\ & H^*(H_1, \mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd} \qquad \begin{tikzcd} H^*(H_1, Z_2) \arrow[rr, "\times 2"] & & H^*(H_1, Z_2) \arrow[dl, "j"] \\ & H^*(H_1, \mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd} . \tag{81}$$

Since all the maps in exact couples are $H^*(H_1, \mathbb{Z})$-module maps, the following proposition completely determines both exact couples.

Proposition 8.3. *In the exact couples (81) differentials $d_1 = j \circ \delta$ are determined, respectively, by*

$$d_1(1) = a, \ d_1(b) = b(b+a) \qquad \textit{and} \qquad d_1(1) = a+b, \ d_1(a) = d_1(b) = ab. \tag{82}$$

Proof. In both claims we use the following diagram of exact couples induced by restrictions, where $i \in \{1, 2\}$,

$$\begin{tikzcd}[column sep=small] & \boxed{\begin{tikzcd} H^*(H_1, Z_i) \arrow[rr, "\times 2"] & & H^*(H_1, Z_i) \arrow[dl, "j"] \\ & H^*(H_1, \mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}} \arrow[dl] \arrow[d] \arrow[dr] & \\ \boxed{\begin{tikzcd} H^*(K_1, Z_i) \arrow[rr, "\times 2"] & & H^*(K_1, Z_i) \arrow[dl, "j"] \\ & H^*(K_1, \mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}} & \boxed{\begin{tikzcd} H^*(K_2, Z_i) \arrow[rr, "\times 2"] & & H^*(K_2, Z_i) \arrow[dl, "j"] \\ & H^*(K_2, \mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}} & \boxed{\begin{tikzcd} H^*(K_3, Z_i) \arrow[rr, "\times 2"] & & H^*(K_3, Z_i) \arrow[dl, "j"] \\ & H^*(K_3, \mathbb{F}_2) \arrow[ul, "\delta"] & \end{tikzcd}} \end{tikzcd}$$

The first exact couple. The module Z_1 is a non-trivial K_1 and K_3-module, but a trivial K_2-module. Therefore by the long exact (53), properties of Steenrod squares and the assumption at the end of the Section 4.3.2:

(A) K_1-exact couple: $d_1(1) = t_1$ and $d_1(t_1) = 0$;
(B) K_2-exact couple: $d_1(1) = 0$ and $d_1(t_2) = t_2^2$;
(C) K_3-exact couple: $d_1(1) = t_3$ and $d_1(t_3) = 0$;

Now

$$\left.\begin{array}{r}\mathrm{res}^{H_1}_{K_1}(d_1(1)) = t_1\\ \mathrm{res}^{H_1}_{K_2}(d_1(1)) = 0\\ \mathrm{res}^{H_1}_{K_3}(d_1(1)) = t_3\end{array}\right\}\Rightarrow d_1(1) = a \qquad \left.\begin{array}{r}\mathrm{res}^{H_1}_{K_1}(d_1(b)) = 0\\ \mathrm{res}^{H_1}_{K_2}(d_1(b)) = t_2^2\\ \mathrm{res}^{H_1}_{K_3}(d_1(b)) = 0\end{array}\right\}\Rightarrow d_1(b) = b(b+a).$$

The second exact couple. The module Z_2 is a non-trivial K_2 and K_3-module, while it is a trivial K_1-module. Therefore by the long exact (53), properties of Steenrod squares and the assumption at the end of the Section 4.3.2:
(A) K_1-exact couple: $d_1(1) = 0$ and $d_1(t_1) = t_1^2$;
(B) K_2-exact couple: $d_1(1) = t_2$ and $d_1(t_2) = 0$;
(C) K_3-exact couple: $d_1(1) = t_3$ and $d_1(t_3) = 0$;
Now

$$\left.\begin{array}{r}\mathrm{res}^{H_1}_{K_1}(d_1(1)) = 0\\ \mathrm{res}^{H_1}_{K_2}(d_1(1)) = t_2\\ \mathrm{res}^{H_1}_{K_3}(d_1(1)) = t_3\end{array}\right\}\Rightarrow d_1(1) = a+b \qquad \left.\begin{array}{r}\mathrm{res}^{H_1}_{K_1}(d_1(b)) = t_1^2\\ \mathrm{res}^{H_1}_{K_2}(d_1(b)) = 0\\ \mathrm{res}^{H_1}_{K_3}(d_1(b)) = 0\end{array}\right\}\Rightarrow d_1(b) = ab.$$

□

Remark 8.4. The result of the previous proposition can be seen as a key step in an alternative proof of the equation (19).

Proposition 8.5. *In the exact couples (79), with identification $H^*(D_8, \mathbb{F}_2[D_8/H_1]) = \mathbb{F}_2[a, a+b]$, the differential $d_1 = j \circ \delta$ satisfies*

$$d_1(1) = a,\ \ d_1(a+b) = d_1(b) = b(b+a),\ \ d_1(a^2) = a^3. \tag{83}$$

(This determines d_1 completely since j and δ are $H^*(D_8, \mathbb{Z})$-module maps.)

Proof. Recall from the Remark 7.5 that the restriction map

$$\mathrm{res}^{D_8}_{H_1} : H^*(D_8, \mathbb{F}_2[D_8/H_1]) \to H^*(H_1, \mathbb{F}_2[D_8/H_1])$$

is injective. Then the equations (83) are obtained by filling the empty places in the following diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{d_1} & \square \\ \downarrow & & \downarrow \\ 1\oplus 1 & \xrightarrow{d_1} & a\oplus(a+b)\end{array} \qquad \begin{array}{ccc} a+b & \xrightarrow{d_1} & \square \\ \downarrow & & \downarrow \\ (a+b)\oplus a & \xrightarrow{d_1} & b(b+a)\oplus ab\end{array} \qquad \begin{array}{ccc} a^2 & \xrightarrow{d_1} & \square \\ \downarrow & & \downarrow \\ a^2\oplus(a+b)^2 & \xrightarrow{d_1} & a^3\oplus(a+b)^3\end{array}$$

where all vertical maps are $\mathrm{res}^{D_8}_{H_1}$. □

Corollary 8.6. *$H^*(D_8, M)$ is generated as a $H^*(D_8, \mathbb{Z})$-module by three elements ζ_1, ζ_2, ζ_3 of degree 1, 2, 3 such that*

$$j(\zeta_1) = a, \qquad j(\zeta_2) = b(a+b), \qquad j(\zeta_3) = a^3$$

where j is the map $H^*(D_8, M) \to H^*(D_8, \mathbb{F}_2[D_8/H_1])$ from the exact couple (79).

8.2 $\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d$

The relation between sequences of polynomials $\pi_d \in H^*(D_8, \mathbb{F}_2)$ and $\Pi_d \in H^*(D_8, \mathbb{Z})$ is described by the following lemma.

Lemma 8.7. *Let $j : H^*(D_8, \mathbb{Z}) \to H^*(D_8, \mathbb{F}_2)$ be the map induced by the coefficient morphism $\mathbb{Z} \to \mathbb{F}_2$ (explicitly given by (39)). Then for every $d \geq 0$,*

$$j(\Pi_d) = \pi_{2d}.$$

Proof. Induction on $d \geq 0$. For $d = 0$ and $d = 1$ the claim is obvious. Let $d \geq 2$ and let us assume that claim holds for every $d \leq k+1$. Then

$$\begin{aligned} j(\Pi_{k+2}) &= j(\mathcal{Y}\Pi_{k+1} + \mathcal{W}\Pi_k) \overset{hypo.}{=} y^2\pi_{2k+2} + w^2\pi_{2k} &&= y^2\pi_{2k+2} + yw\pi_{2d+1} + yw\pi_{2d+1} + w^2\pi_{2k} \\ &= y(y\pi_{2k+2} + w\pi_{2d+1}) + w(y\pi_{2d+1} + w\pi_{2k}) &&= y\pi_{2k+3} + w\pi_{2k+2} \\ &= \pi_{2k+4}. \end{aligned}$$

□

There is a sequence of D_8-inclusions

$$S^1 \times S^1 \subset S^2 \times S^2 \subset ... \subset S^{d-1} \times S^{d-1} \subset S^d \times S^d \subset S^{d+1} \times S^{d+1} \subset ...$$

implying a sequence of ideal inclusions

$$\mathrm{Index}^3_{D_8,\mathbb{Z}} S^1 \times S^1 \supseteq \mathrm{Index}^4_{D_8,\mathbb{Z}} S^2 \times S^2 \supseteq ... \supseteq \mathrm{Index}^{d+1}_{D_8,\mathbb{Z}} S^{d-1} \times S^{d-1} \supseteq \mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d \supseteq ... \tag{84}$$

8.2.1 The case when d is odd

In this section we prove that

$$\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d = \langle \Pi_{\frac{d+1}{2}}, \Pi_{\frac{d+3}{2}} \rangle. \tag{85}$$

The proof can be conducted as in the case of $\mathbb{F}_2$ coefficients (Section 7.2). The results of the section 7.2 can also be used to simplify the proof of equation (85). The morphism $j : H^*(D_8, \mathbb{Z}) \to H^*(D_8, \mathbb{F}_2)$ induced by the coefficient morphism $\mathbb{Z} \to \mathbb{F}_2$ is a part of the morphism J of Serre spectral sequences (77) and (60). Thus, for $1 \in E^{0,d}_{d+1} = H^0(D_8, H^d(S^d \times S^d, \mathbb{Z}))$, $\hat{1} \in E^{0,d}_{d+1} = H^0(D_8, H^d(S^d \times S^d, \mathbb{F}_2))$, $\alpha \in E^{2,d}_{d+1} = H^2(D_8, H^d(S^d \times S^d, \mathbb{Z}))$ and $a \in E^{1,d}_{d+1} = H^1(D_8, H^d(S^d \times S^d, \mathbb{Z}))$,

$$\begin{aligned} J(\partial_{d+1}(1)) &= \partial_{d+1}(J(1)) = \partial_{d+1}(\hat{1}) = \pi_{d+1} = J\left(\Pi_{\frac{d+1}{2}}\right), \\ J(\partial_{d+1}(\alpha)) &= \partial_{d+1}(J(\alpha)) = \partial_{d+1}(a^2) = \partial_{d+1}(w \cdot \hat{1} + y \cdot a) = w\pi_{d+1} + y\pi_{d+2} = \pi_{d+3} = J\left(\Pi_{\frac{d+3}{2}}\right). \end{aligned}$$

From Proposition 8.2 and the sequence of inclusions (84) it follows that

$$\partial_{d+1}(1) = \Pi_{\frac{d+1}{2}} \text{ and } \partial_{d+1}(\alpha) = \Pi_{\frac{d+3}{2}}.$$

Finally, the statement (B) of Proposition 8.2 implies equation (85).

8.2.2 The case when d is even

In this section we prove that

$$\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d = \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}, \mathcal{M}\Pi_{\frac{d}{2}} \rangle. \tag{86}$$

The previous section implies that

$$\langle \Pi_{\frac{d}{2}}, \Pi_{\frac{d+2}{2}} \rangle \supseteq \mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d \supseteq \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}} \rangle. \tag{87}$$

From Corollary 8.6 we know that $\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d$ is generated by three elements $\partial_{d+1}(\zeta_1)$, $\partial_{d+1}(\zeta_2)$, $\partial_{d+1}(\zeta_3)$ of degrees $d+2$, $d+3$, $d+4$. Thus, $\partial_{d+1}(\zeta_1) = \Pi_{\frac{d+2}{2}}$ and $\partial_{d+1}(\zeta_2) = \mathcal{M}\Pi_{\frac{d}{2}}$. Since $\Pi_{\frac{d+4}{2}} \notin \langle \Pi_{\frac{d+2}{2}}, \mathcal{M}\Pi_{\frac{d}{2}} \rangle$, then $\partial_{d+1}(\zeta_3) = \Pi_{\frac{d+4}{2}}$. The proof of the equation (86) is concluded.

Alternatively, the proof can be obtained with the help of the morphism J of Serre spectral sequences (77) and (60).

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